Linear Quadratic Mean Field Social Control with Common Noise: A Directly Decoupling Method

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Abstract

This paper is concerned with mean field linear quadratic social control with common noise, where the weight matrices of individual costs are indefinite. We first obtain a set of forward-backward stochastic differential equations (FBSDEs) from variational analysis, and then, construct a centralized feedback representation by decoupling the FBSDEs. By using solutions of two Riccati equations, we design a set of decentralized control laws, which is further shown to be asymptotically social optimal. The necessary and sufficient conditions are given for uniform stabilization of the systems by exploiting the relation between population state average and aggregate effect. An explicit expression of the optimal social cost is given in term of two Riccati equations. Besides, the decentralized optimal solution is provided for mean field social control problem with finite agents.

Key words: Mean field game, optimal social cost, stabilization, finite agents, common noise, FBSDE

1 Introduction

1.1 Background and Motivation

The subject of mean field (MF) games and control has attracted increasing attention from communities of system control, mathematics and economics [4], [5], [12], [7]. A MF model involves a large number of interactive players (agents), where a key feature is while the influence of each agent is negligible, the impact of the overall population is significant to each one. The main idea of MF approximations is to replace the average interaction of an agent with all other agents by aggregation effect. By consistent MF approximations, the dimensionality difficulty is overcome. By now, the linear quadratic (LQ) framework has been commonly adopted in MF studies because of its analytical tractability and close connection to practical applications. In this aspect, some relevant works include [20], [29], [10], [41], [3], [31]. Huang et al. developed the Nash certainty equivalence (NCE) based on the fixed-point analysis and designed $\epsilon$-Nash equilibria for large-population LQ games with discounted costs [20]. The NCE approach was then applied to the more general cases with ergodic costs [29] and with Markov jump parameters [41], respectively. The works [6], [3] employed the adjoint equation approach and the fixed-point theorem to obtain sufficient conditions for the existence of equilibrium strategies over a finite horizon. For other aspects of MF games, readers are referred to [22], [27], [6] for nonlinear MF games, [45] for oblivious equilibrium in dynamic games, [19], [42], for MF games with major players, [16], [31] for robust MF games.

Mean field games with common noise are a kind of large-population games with correlated players who share a common random noise [8]. The common noise may be interpreted as a passive version of the major player [18]. Such modeling can accommodate considerable situations since common noise may represent some external factors with influence on all players. This is well-framed in reality, particularly in finance and economics; for instance, the physical environment for all particles or a financial policy for all market participants [14], [9]. Due to the impact of common noise, all the players are not independent of each other, but correlative or strong coupling. Consequently, MF games with common
noise are within a more general setting, and the related analysis becomes more technical. The work [8] studied strong and weak solutions to MF games with common noise from the fixed point analysis in random measure flows. The wellposedness problem was further considered in [1]. The work [13] investigated a general LQ-MF type control and connected it to mean field games with common noise. The MF limit of large-population symmetric games was derived with and without common noise, respectively [28]. Besides noncooperative games, the social optima in MF models with weak coupling have drawn more research interests. By social optimization, all players in a large-population system cooperate to optimize the common social cost—the sum of individual costs. Accordingly, we formulate a type of team decision problem [35]. Different from Nash games, all players in a team problem share the same cost and cooperate to reach a social optimum although they may have different information sets [15]. The work [21] studied social optima in MF-LQ control, and provided an asymptotic team-optimal solution. Subsequently, [43] investigated a MF social optimal problem where the Markov jump parameter appears as a common source of randomness. The work [23] designed socially optimal strategies by analyzing forward backward stochastic differential equations (FBSDEs). In [36], authors investigated dynamic cooperative collective choice by finding a social optimum. Furthermore, some recent studies focused on stochastic dynamic teams and their MF limit, such as [37].

1.2 Novelty and Contributions

This paper investigates decentralized control of MF-LQ social control with common noise, where the weight matrices in individual costs are indefinite. The presence of common noise leads to the strong correlations of all agents, which makes the corresponding analysis much more complicated [8]. Most previous works on MF social control applied the fixed-point method and person-by-person optimality (see e.g. [21], [43], [36]). However, due to the appearance of common noise, the corresponding fixed-point analysis is very complicated [17]. In this paper, we obtain decentralized control of the problem by decoupling directly high-dimensional FBSDEs with MF approximations. This procedure shares a similar philosophy with the direct method [24], [40], [27]. In recent years, some progress has been made for the study of the optimal LQ control by tackling FBSDEs. Readers are referred to [46], [6], [50], [40], [19] etc. for details.

For the finite-horizon problem, we first obtain a set of high-dimensional FBSDEs by tackling the large-scale social control problem, and give a centralized feedback-type control laws (only depending on the state of a representative agent and population state average) by decoupling the FBSDEs. By the MF heuristics, we design a set of decentralized control laws, which is further shown to be asymptotically optimal. An explicit form of the asymptotic social optimal cost is given in terms of solutions to two Riccati equations. Besides, we apply the result to obtain decentralized solution to a class of MF social control problem with finite agents. For the infinite-horizon case, we first design a set of decentralized control laws with help of two Riccati equations, and then, show the consistency of MF approximations by the Lyapunov function method. By exploiting the relationship between population state average and aggregate effect, two equivalent conditions are given for uniform stabilization of all the subsystems.

The main contributions of the paper are listed as follows.

- For the finite-horizon problem, we first obtain the existence conditions of centralized optimal control by variational analysis, and then, design a feedback decentralized control by decoupling FBSDEs and applying MF approximations.
- With the help of condition expectation, we give the decentralized optimal solution to a class of MF social control problem with finite agents.
- For the infinite-horizon problem, feedback decentralized control laws are designed with help of the solutions to Riccati equations. Two equivalent conditions are given for uniform stabilization of all the subsystems by making full use of the relation between population state average and aggregate effect.
- Under some basic assumptions we obtain asymptotic optimality of decentralized control. No fixed-point equations or additional assumptions are needed. Besides, an explicit expression of the optimal social cost is given by virtue of two Riccati equations.

1.3 Organization and Notation

The organization of the paper is as follows. In Section 2, we first design the asymptotical optimal control for finite-horizon MF social control problems, and give the corresponding social cost. Besides, we give the optimal decentralized control of MF social problems with finite agents. In Section 3, we design decentralized controls for the infinite-horizon case and further give necessary and sufficient conditions for uniform stabilization of the systems. In Section 4, a numerical example is given to verify the results. Section 5 concludes the paper.

The following notation will be used throughout this paper. We use $|\cdot|$ to denote the norm of a Euclidean space, or the Frobenius norm for matrices. For a symmetric matrix $Q$ and a vector $z$, $|z|^2_Q = z^T Q z$, $Q^\dagger$ is the Moore-Penrose pseudoinverse of the matrix $Q$. $R(Q)$ denotes the range of a matrix (or an operator) $Q$. For a vector or matrix $M$, $M^T$ denotes its transpose, and $M > 0$ ($M \geq 0$) means that $M$ is positive definite (nonnegative definite). Let $\mathbb{R}^n$ denote the space of all symmetric $n \times n$ matrices. Let $L^\infty(0, T; \mathbb{R}^{n \times k})$ (resp., $C([0, T]; \mathbb{R}^{n \times k})$) be the space of all $\mathbb{R}^{n \times k}$-valued bounded (resp. continuous bounded) functions; let $L^2_2(0, T; \mathbb{R}^{k})$ be the space of all $\mathbb{R}^k$-valued $\mathcal{F}_t$-adapted processes $x(\cdot)$ satisfying

$$Q^\dagger$$ is a unique matrix satisfying $QQ^\dagger Q = Q^\dagger, Q^\dagger QQ^\dagger = Q, (Q^\dagger Q)^T = Q^\dagger Q$, and $(QQ^\dagger)^T = QQ^\dagger$. 

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Consider a large population system with $N$ agents. The $i$th agent, $\mathcal{A}_i$, evolves by the following stochastic differential equation (SDE):

$$
dx_t(x_i(t)) = [A(t)x_i(t) + G(t)x^{(N)}(t) + B(t)u_i(t) + f(t)]dt + [C(t)x_i(t) + D(t)u_i(t) + \sigma(t)]dW_i(t) + [C_0(t)x_i(t) + D_0(t)u_i(t) + \sigma_0(t)]dW_0(t),
$$

where $1 \leq i \leq N$, $x_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}^r$ are the state and input of the agent $\mathcal{A}_i$. $x^{(N)}(t) = \frac{1}{N}\sum_{j=1}^{N} x_j(t)$ is the population state average. $\{W_i(t), 0 \leq i \leq N\}$ are a sequence of independent 1-dimensional Brownian motions on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$. $A, B, G, C, D, C_0$ and $D_0$ are deterministic matrix-valued functions with compatible dimensions, where $\mathcal{F}_t = \sigma(W_0(s), 0 \leq s \leq t)$, $f, \sigma$ and $\sigma_0$ are $\mathcal{F}_t$-adapted vector-valued processes in $\mathbb{R}^n$, reflecting the impact on each agent by the environment. The cost functional of $\mathcal{A}_i$ is given by

$$
J_i^F(u) = \mathbb{E}\int_0^T \left[ x_i(t) - \Gamma(t)x^{(N)}(t) - \eta(t) \right]^2 Q(t) + |u_i(t)|^2 R(t) dt + \mathbb{E}[x_T - \Gamma_0 x^{(N)}(T) - \eta_T]^2 H.
$$

where $Q, R$ and $\Gamma$ are bounded deterministic matrix-valued functions with compatible dimensions. Let $u = \{u_1, \ldots, u_i, \ldots, u_N\}$. Let $\mathcal{F}_i = \sigma(x_i(0))$, $\mathcal{G}_i = \sigma(\mathcal{F}_i)$. Define the centralized control set as

$$
\mathcal{U}_c = \{ (u_1, \ldots, u_N) | u_i(t) \text{ is adapted to } \mathcal{F}_i, \mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty \},
$$

and the decentralized control set as

$$
\mathcal{U}_d = \{ (u_1, \ldots, u_N) | u_i(t) \text{ is adapted to } \mathcal{G}_i, \mathbb{E}\int_0^T |u_i(t)|^2 dt < \infty \}.
$$

In this section, we mainly study the two problems:

(P1). Seek a set of centralized control laws to optimize the social cost $J_{soc}^F$ for the system (1)-(2), i.e., $\inf_{u \in \mathcal{U}_c} J_{soc}^F(u)$, where $J_{soc}^F(u) = \sum_{i=1}^N J_i^F(u)$.

(P1'). Seek a set of decentralized controls to optimize social cost $J_{soc}^F$ for the system (1)-(2), i.e., $\inf_{u \in \mathcal{U}_d} J_{soc}^F(u)$. Assume

A1) The coefficients satisfy the following conditions:

(i) $A, G, C, C_0, \Gamma \in L^\infty(0, T; \mathbb{R}^{n \times n})$, and $B, D, D_0 \in L^\infty(0, T; \mathbb{R}^{n \times r})$;

(ii) $Q \in L^\infty(0, T; \mathbb{S}^n)$, and $R \in L^\infty(0, T; \mathbb{S}^r)$;

(iii) $f, \sigma, \sigma_0, \eta \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$;

(iv) $\Gamma_0 \in \mathbb{R}^{n \times n}$, and $H \in \mathbb{S}^n$ are bounded; $\eta_0 \in \mathbb{R}^n$ is $\mathcal{F}_t$-adapted, and $\mathbb{E}[|\eta_0|^2] < \infty$.

A2) $x_i(0), i = 1, \ldots, N$ are mutually independent and have the same mathematical expectation, $E[x_i(0)] = x_0$, $i = 1, \ldots, N$. There exists a constant $c_0$ (independent of $N$) such that $\max_{1 \leq i \leq N} E[|x_i(0)|^2] < c_0$.

We now provide a practical example.

Example 2.1 (Systemic risk model). Consider a model of inter-bank borrowing and lending where the log-monetary reserves $x_i$, $i = 1, \ldots, N$ evolve by

$$
dx_t(x_i(t)) = -A(t)(x_i(t) - x^{(N)}(t))dt + B(t)dt + \sigma(t)dW_i(t) + \sigma_0(t)dW_0(t),
$$

where $W_0$ is a common Brownian motion, and $W_i, i = 1, \ldots, N$ are idiosyncratic Brownian motions, independent of $W_0$. $A(t)$ is the rate of mean-reversion in interactions from inter-bank borrowing and lending; $\sigma(W_0)$ is stochastic volatility of bank reserves. $\rho \in [-1, 1]$ is the correlation coefficient between idiosyncratic noise and common noise. Each bank controls its rate of lending or borrowing to minimize $J_{soc}^F(u) = \sum_{i=1}^N J_i^F(u)$ where

$$
J_i^F(u) = \mathbb{E}\int_0^T \left[ Q(t)x_i(t) - x^{(N)}(t) \right]^2 dt + u_i^2(t) dt.
$$

Compared with the original model in [9], social optimization is considered here with applications in government regulation, community welfare, etc.

From now on, time variable $t$ may be suppressed if no confusion occurs. To simplify the statement, denote by

$$
\begin{aligned}
Q_t &\triangleq \Gamma^T Q \Gamma - \Gamma^T Q \Gamma_t, \\
H_t &\triangleq \Gamma_0^T H \Gamma_t - \Gamma_0^T H \Gamma_0, \\
\eta &\triangleq \Gamma^T Q \eta, \\
\eta_0 &\triangleq \Gamma_0^T H \eta_0 - \Gamma_0^T H \eta_0.
\end{aligned}
$$

2.1 Design of control laws

Definition 2.1 ([48]) Problem (P1) is convex, if for any $0 < \lambda < 1$ and $u, v \in L^2_{\mathbb{F}}(0, T; \mathbb{R}^r)$,

$$
J_{soc}^F(\lambda u + (1 - \lambda)v) \leq \lambda J_{soc}^F(u) + (1 - \lambda)J_{soc}^F(v).
$$

Particularly, (P1) is uniformly convex, if we have

$$
J_{soc}^F(\lambda u + (1 - \lambda)v) \leq \lambda J_{soc}^F(u) + (1 - \lambda)J_{soc}^F(v) - \lambda(1 - \lambda)\mathbb{E}\int_0^T |u - v|^2 dt.
$$
We first obtain the necessary and sufficient conditions for the existence of centralized optimal control of (P1).

**Theorem 2.1** Assume A1)-A2) hold. Then, (P1) has an optimal control \( \tilde{u} \in \mathcal{U}_t \) if and only if (P1) is convex in \( u \) and the following equation system admits a set of solutions \((\bar{x}_i, p_i, (\bar{\beta}^{(i)}_j))_{j=1}, i = 1, \ldots, N\):

\[
\begin{aligned}
d\bar{x}_i &= \left(A_{\bar{x}_i} \bar{x}_i + B_\mu \bar{u}_i + G \bar{x}^{(N)} + \varphi \right) dt + (C_{\bar{x}_i} \bar{x}_i + D_\mu \bar{u}_i + \sigma) dW_i \\
&\quad + (C_0 \bar{x}_i + D_0 \bar{u}_i + \sigma_0) dW_0, \quad \bar{x}_i(0) = x_{i0}, \\
dp_i &= -\left(A^T \bar{p}_i + G^T \bar{p}^{(N)} + C_0^T \bar{\beta}_i^0 + C^T \bar{\beta}_i^1 \right) dt \\
&\quad - (Q_{\bar{x}_i} - \bar{Q} \bar{x}^{(N)} - \bar{\eta}) dt + \sum_{j=0}^N \bar{\beta}_i^j dW_j, \\
p_i(T) &= H_{\bar{x}_i}(T) - H_{\bar{u}_i}(T) - \bar{\eta}_i, \quad i = 1, \ldots, N,
\end{aligned}
\]

where \( p_i^{(N)} = \frac{1}{N} \sum_{i=1}^N p_i \) and the optimal control \( \bar{u}_i \) satisfies the stationary condition

\[
R \bar{u}_i + B^T \bar{p}_i + D^T \bar{\beta}_i^0 + D_0^T \bar{\beta}_i^1 = 0.
\]

Particularly, if Problem (P1) is uniformly convex, then (P1) admits an optimal control necessarily.

**Proof.** See Appendix A. \( \square \)

Equations (3)-(4) are a fully coupled FBSDE. It is nonstandard due to MF terms appearing in both forward and backward equations, thus its solvability becomes quite technical. Related works on solvability and control problems of FBSDEs may be referred to [47], [32], [44].

Now we make the following assumption:

**A3** Problem (P1) is uniformly convex. \(^2\)

By A3), (P1) admits a unique optimal control. The next step is to obtain a proper form for deriving the decentralized feedback representation of optimal control.

Denote \( \bar{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N \bar{x}_i, \bar{x}^{(N)}_{i0} = \frac{1}{N} \sum_{i=1}^N \bar{x}_{i0} \) and \( \bar{u}^{(N)} = \frac{1}{N} \sum_{i=1}^N \bar{u}_i \). It follows from (3) that

\[
d\bar{x}^{(N)} = \left[ (A + G) \bar{x}^{(N)} + B \bar{u}^{(N)} + \varphi \right] dt \\
\quad + \frac{1}{N} \sum_{i=1}^N (C_{\bar{x}_i} \bar{x}_i + D_\mu \bar{u}_i + \sigma) dW_i \\
\quad + (C_0 \bar{x}^{(N)} + D_0 \bar{u}^{(N)} + \sigma_0) dW_0, \quad \bar{x}^{(N)}(0) = \bar{x}^{(N)}_{i0}. \tag{5}
\]

We now use the idea proposed by [50], [47] to decouple the FBSDE (3). Let \( p_i = P_i \bar{x}_i + K_i \bar{x}^{(N)} + \varphi_i, \) where \( P_i, K_i \in \mathbb{R}^{n \times n} \), and \( \varphi_i \) satisfies

\[
d\varphi_i = \varphi_i dt + \gamma_i dW_0, \quad \varphi_i(T) = -\bar{\eta}_i.
\]

Here \( \varphi_i \) and \( \gamma_i \) are to be determined. Then, by (3), (5) and Itô’s formula,

\[
\begin{aligned}
dp_i &= P_i \left[ (A_{\bar{x}_i} + B_{\bar{u}_i} + G \bar{x}^{(N)} + \varphi) dt + (C_{\bar{x}_i} + D_\mu \bar{u}_i + \sigma) dW_i \\
&\quad + (C_0 \bar{x}_i + D_0 \bar{u}_i + \sigma_0) dW_0 \right] \\
&\quad + K_i \left[ (A + G) \bar{x}^{(N)} + B \bar{u}^{(N)} + \varphi \right] dt \\
&\quad + \frac{1}{N} \sum_{j=1}^N (C_{\bar{x}_j} \bar{x}_j + D_\mu \bar{u}_j + \sigma) dW_j \\
&\quad + (C_0 \bar{x}^{(N)} + D_0 \bar{u}^{(N)} + \sigma_0) dW_0 \right] dt \\
&\quad + \varphi_i dt + \gamma_i dW_0
\end{aligned}
\]

Comparing this with (3), it follows that

\[
\begin{aligned}
\beta_i^0 &= P_i (C_{\bar{x}_i} \bar{x}_i + D_\mu \bar{u}_i + \sigma_0) \\
&\quad + K_i (C_0 \bar{x}^{(N)} + D_0 \bar{u}^{(N)} + \sigma_0) + \gamma_i, \\
\beta_i^1 &= (P_i + \frac{1}{N} K_i) (C_{\bar{x}_i} + D_\mu \bar{u}_i + \sigma), \\
\beta_i^0 &= \frac{1}{N} K_i (C_{\bar{x}_j} + D_\mu \bar{u}_j + \sigma), \quad 1 \leq j \neq i \leq N.
\end{aligned}
\]

From (4), we have for any \( i = 1, \ldots, N, \)

\[
\begin{aligned}
T \varphi_i + \Psi \bar{x}_i + \Theta \bar{x}^{(N)} + D_0^T K_i D_0 (\bar{u}^{(N)} + \psi_i) = 0,
\end{aligned}
\]

where

\[
\begin{aligned}
T &= R + D^T (P_i + \frac{K_i}{N}) D + D^T P_i D_0, \\
\Psi &= B^T P_i + D^T (P_i + \frac{1}{N} K_i) C + D^T P_i C_0, \\
\Theta &= B^T K_i + D^T K_i C_0, \\
\psi_i &= D^T (P_i + \frac{K_i}{N}) \sigma + D^T (P_i + K_i) \sigma_0 \\
&\quad + B^T \varphi_i + D_0^T \gamma_i.
\end{aligned}
\]

This further implies

\[
\begin{aligned}
\bar{u}^{(N)} &= -((T \varphi_i + \Psi \bar{x}_i + \Theta \bar{x}^{(N)} + D_0^T K_i D_0) + [(\Psi + \Theta) \bar{x}^{(N)} + \psi_i].
\end{aligned}
\]

Thus, we obtain that the optimal control is given by

\[
\begin{aligned}
\bar{u}_i &= -T \varphi_i + \Psi \bar{x}_i + \Theta \bar{x}^{(N)} + D_0^T K_i D_0 \\
&\quad \times ((\Psi + \Theta) \bar{x}^{(N)} + \psi_i).
\end{aligned}
\]

This together with (6) gives

\[
\begin{aligned}
\hat{P}_N &+ A^T P_N + P_N A + C^T (P_N + \frac{K_i}{N}) C + Q \\
&+ C_0^T P_N C_0 - \Psi \hat{T}_N \Psi = 0, \quad P_N(T) = H, \\
\hat{K}_N &+ (A+G)^T K_N + K_N (A+G) + C_0^T K_N C_0 + C^T P_N \\
&+ P_N G - (\Psi + \Theta)^T \hat{T}_N + D_0^T K_N D_0 \psi_i (\Psi + \Theta) \\
&+ \Psi \hat{T}_N \Psi - Q_T = 0, \quad K_N(T) = -H_{\gamma_i}.
\end{aligned}
\]

\(^2\) By [38, Theorem 4.5], a necessary and sufficient condition to guarantee A3) is that the corresponding Riccati equation admits a strongly regular solution.
For further analysis, we assume \( N \to \infty \). By the law of large numbers, we may approximate \( x^{(N)} \) in (5) with \( \bar{x} \), which satisfies

\[
d\bar{x} = \left\{ [A + G - B\bar{T}^1(\Psi + \Theta)]\bar{x} - B\bar{T}^1\psi + f \right\} dt + \left\{ [C_0 - D_0\bar{T}^1(\Psi + \Theta)]\bar{x} - D_0\bar{T}^1\psi + \sigma_0 \right\} dW_0,
\]

with \( \bar{x}(0) = \bar{x}_0 \). By Proposition 2.1, the decentralized control law for agent \( A_i \), \( i = 1, \ldots, N \) may be taken as

\[
\hat{u}_i(t) = -\bar{T}_i(t)[\Psi(t)(x_i(t) - \bar{x}(t))]
- \bar{T}_i(t)[(\Psi(t) + \Theta(t))\bar{x}(t) + \psi(t)],
\]

where \( \bar{T}, \hat{T}, \Psi, \Theta, \psi \) and \( \bar{x} \) are determined by (11)-(15), and \( \bar{x}_i \) satisfies

\[
dx_i = \left\{ [\hat{A}\hat{x}_i + G\hat{x}^{(N)}] + B[\hat{T}^1\Psi - \bar{T}^1(\Psi + \Theta)]\bar{x} - B\bar{T}^1\psi + f \right\} dt + \left\{ C_i\hat{x}_i + D_i\bar{T}^1\Psi \right.
- \bar{T}^1(\Psi + \Theta)\bar{x} - D_0\bar{T}^1\psi + \sigma_0 \right\} dW_i,
\]

with

\[
\hat{A} \triangleq A - B\bar{T}\Psi, \quad \hat{C} \triangleq C - D\bar{T}^1\Psi, \quad \hat{C}_0 \triangleq C_0 - D_0\bar{T}^1\Psi.
\]

**Remark 2.2** Previous works (e.g., [21], [6], [3], [13]) considered LQ-MF models by the fixed-point approach. To achieve asymptotic optimality, an additional condition like the solvability of fixed point equations is needed, which is not easy to be verified. Furthermore, for MF social control with common noise, the tackling process is more complicated. [17] and [33] first constructed a new auxiliary system by two-step duality, and then derived a consistency condition system, which is a MF FBSDE with embedding representation. Here, we get rid of fixed-point conditions and MF FBSDEs thoroughly (Note that \( \varphi \) and \( \bar{x} \) are fully decoupled).

We are in a position to give asymptotic social optimality of the decentralized control.

**Theorem 2.2** Assume that A1)-A4) hold. For Problem (P'), the set of decentralized control laws \( \{ \hat{u}_1, \cdots, \hat{u}_N \} \) given by (16) has asymptotic social optimality, i.e.,

\[
\left| \frac{1}{N} J_{soc}^{F}(\hat{u}) - \frac{1}{N} \inf_{u \in U_c} J_{soc}^{F}(u) \right| = O\left( \frac{1}{\sqrt{N}} \right).
\]

Proof. See Appendix B.}

2.2 Asymptotically social optimal cost

We now give an explicit expression of the asymptotic average social cost in terms of two Riccati equations.

**Theorem 2.3** Assume that A1)-A4) hold and \( \{ x_0 \} \) have the same variance. Then, for (P'), the asymptotic
average social optimum is given by
\[
\lim_{N \to \infty} \frac{1}{N} J_{soc}^N(u) = \mathbb{E}[(x_{i0} - \bar{x}_0)^T P(0)(x_{i0} - \bar{x}_0) + \bar{x}_0^T \Pi(0) \bar{x}_0 + 2 \varphi^T(0) \bar{x}_0 + q_T],
\]
where \( P, \Pi \) are given by (12) and (13) (\( \Pi = P + K \)), and
\[
q_T := \mathbb{E} \int_0^T \left[ |\sigma|^2 + |\sigma_0|^2 + 2 \varphi^T f + 2 \gamma^T \rho_0 + |\eta|^2 \right] dt + \mathbb{E} [||\eta||_H^2].
\]

Proof. See Appendix C. \( \square \)

**Remark 2.3** The MF-type control problem (see e.g., [13], [17]) is a closely related problem to the MF social control. The setups of both problems are different. The MF social control is multi-agent optimization, and there are a large number of agents with their own states and controls. In contrast, the mean field type control is single-agent optimization. However, the optimal solutions of both problems coincide for the infinite population case (See details in Appendix C).

2.3 Mean field social optimal control with finite agents

We now consider the case \( G = 0 \). Denote \( H_i^* = \sigma(x_i(0), W_i(s), W_0(s), s \leq t) \) and
\[
\mathcal{U}_x^* = \left\{ \{u_1, \ldots, u_N\} | u_i(t) \text{ is adapted to } H_i^*, \mathbb{E} \int_0^T |u_i(t)|^2 dt < \infty \right\}.
\]

In analogy to Theorem 2.1, we obtain the result for the existence of decentralized optimal control of (P1').

**Proposition 2.2** Assume that A1)-A2) hold and (P1') is convex in \( u \). Then, \( (u_i^*, i = 1, \ldots, N) \) is a set of optimal control laws of (P1') with respect to \( \mathcal{U}_x^* \) if and only if the following equation admits a set of solutions \((x_i^*, p_i^*, \beta_i^*, \gamma_i^*)\) for \( i = 1, \ldots, N\): \(j = 0, \ldots, N\):
\[
\begin{align*}
    dx_i^* &= (Ax_i^* + Bu_i^* + f)dt + (Cx_i^* + Du_i^* + \sigma)dw_i \\
    dp_i^* &= -[ATp_i^* + C_T \beta_i^* + C^T \gamma_i^*]dt \\
    &\quad - (Qx_i^* - QT x_i^{(N)} - \eta)dt + \sum_{j=0}^N \beta_i^{(j)}dw_j, \\
    p_i(T) &= H x_i^*(T) - H_{r_i} x_i^{(N)}(T) - \eta_0, \quad i = 1, \ldots, N,
\end{align*}
\]
where \( x_i^{(N)} = \frac{1}{N} \sum_{j=1}^N x_i^* \) and the optimal control \( u_i^* \) satisfies the stationarity condition
\[
Ru_i^* + \mathbb{E}[B^T p_i^* + D^T \beta_i^* + D_0^T \beta_0^{(i)}|H_i^*] = 0.
\]

Proof. See Appendix D. \( \square \)

**Remark 2.4** The main difference between Theorem 2.1 and Proposition 2.2 lies in different stationarity conditions. Intuitively, (20) is obtained by applying conditional expectation into the zero variation condition.

Note that \( W_i, W_j \) \( j \neq i \) and \( W_0 \) are mutually independent, and \( x_i^*(t) \in H_i^* \). We have
\[
\mathbb{E}[x_j^*(t)|H_i^*] = \mathbb{E}[x_j^*(t)|F^N_0] = \mathbb{E}[x_j^*(t)], \quad j \neq i.
\]

It follows from (19) that
\[
\begin{align*}
    d \mathbb{E}[x_j^*] &= (A \mathbb{E}[x_j^*] + B \mathbb{E}[u_j^*] + f)dt \\
    &\quad + (C_0 \mathbb{E}[x_j^*] + D_0 \mathbb{E}[u_j^*] + \sigma_0)dw_0, \\
    d \mathbb{E}[p_j^*] &= -[A^T \mathbb{E}[p_j^*] + C_T^* \mathbb{E}[\gamma_j^*] + (Q - \frac{1}{N} Q_T) x_j^* - \bar{\eta}]dt \\
    &\quad - \frac{N-1}{N} Q_T \mathbb{E}[x_j^*]dt + \beta_0^{(j)*} dw_0 + \beta_j^{(j)*} dw_j, \\
    x_j(0) &= x_{i0}, \quad \mathbb{E}[p_j(T)] = (H - \frac{1}{N} H_T) x_j^*(T) \\
    &\quad - H_{r_j} \mathbb{E}[x_j^*(T)] - \bar{\eta}_0, \quad i = 1, \ldots, N.
\end{align*}
\]
Define
\[
\begin{align*}
    u_j^* &= -\tilde{\Psi}_N^+ \Psi_N (x_j^* - \mathbb{E}[x_j^*]) - (\tilde{\Psi}_N + D_0^T K_N D_0)^+ \\
    &\times [(\Psi_N + \Theta_N) \mathbb{E}[x_j^*] + \psi_N],
\end{align*}
\]
where \( \tilde{\Psi}_N, \Psi_N, \Theta_N \) are given by
\[
\begin{align*}
    \tilde{\Psi}_N &= R + D^T P_N^* D + D_0^T P_N^* D_0, \\
    \Psi_N &= B^T P_N^* + D^T P_N^* C + D_0^T P_N^* C_0, \\
    \Theta_N &= B^T K_N^* + D_0^T K_N^* C_0, \\
    \psi_N &= D^T P_N^* \sigma + D_0^T (P_N^* + K_N^*) \sigma_0 + B^T \varphi_N + D_0^T \gamma_N, \\
\end{align*}
\]
and \( P_N^*, K_N^* \) and \( \varphi_N \) satisfy
\[
\begin{align*}
    &\tilde{P}_N^* + A^T \tilde{P}_N^* + \tilde{P}_N^* A + C^T \tilde{P}_N^* C + C_0^T \tilde{P}_N^* C_0 \\
    &\quad + Q - \frac{1}{N} Q_T - \tilde{\Psi}_N^+ \tilde{\Psi}_N = 0, \quad P_N^*(T) = H, \\
    &\tilde{K}_N^* + A^T \tilde{K}_N^* + \tilde{K}_N^* A + C_0^T \tilde{K}_N^* C_0 + \tilde{\Psi}_N^+ \tilde{\Psi}_N = 0, \\
    &\tilde{\Psi}_N + \Theta_N)^T (\tilde{\Psi}_N^+ + \tilde{\Psi}_N)^T \tilde{\Psi}_N, \\
    &\quad \frac{N-1}{N} Q_T = 0, \quad K_N(T) = -H_{r_N}, \\
    &\frac{d \varphi_N}{dt} = -\left[ A - B(\tilde{\Psi}_N + D_0^T K_N^* D_0)^+ (\tilde{\Psi}_N + \Theta_N)^T \right] P_N^* \varphi_N \\
    &\quad + [C - D(\tilde{\Psi}_N + D_0^T K_N^* D_0)^+ (\tilde{\Psi}_N + \Theta_N)^T] P_N^* \sigma \\
    &\quad + \left[ C_0 - D_0(\tilde{\Psi}_N + D_0^T K_N^* D_0)^+ (\tilde{\Psi}_N + \Theta_N)^T \right] \psi_N \\
    &\quad + [(P_N^* + K_N^*) \sigma_0 + \gamma_N^+ + (P_N^* + K_N^*) \varphi_N - \bar{\eta}] dt \\
    &\quad \left[ \gamma_N \right] dw_0, \quad \varphi_N(T) = -\bar{\eta}_N.
\end{align*}
\]
Theorem 2.4 Assume that $A(t) - A(0)$ hold and $(P'(t))$ is convex in $u$. Equations (8)-(9) admit a set of solutions. Then, $(u_0, \ldots, u_N)$ given by (23) is social optimal with respect to the decentralized control set $\mathcal{U}_i$, i.e., for any $u \in \mathcal{U}_i$, the following holds: $J^F_{soc}(u_i) \leq J^F_{soc}(u)$. 

Proof. See Appendix D. \[ \square \]

Remark 2.5 In general, there are two methods to tackle Problem $(P'(t))$. The first one is to find asymptotically optimal solutions; another one is to seek the optimal solution for $(P')$ with $N$ agents. Most works on MF control focused on the former one. Indeed, the latter one can be obtained in some cases. Theorem 2.4 provides a social optimal solution for $(P')$ with $N$ agents.

3 Infinite Horizon Mean Field LQ Control

In this section, we consider the infinite-horizon MF social control problem. For simplicity, suppose $A(t), B(t), C(t), D(t), C_0(t), D_0(t), \Gamma(t), Q(t)$ and $R(t)$ are constant matrices. Assume $Q \succeq 0$ and $R > 0$; $f, \sigma, \sigma_0, \eta \in L^2_{\mathbb{R}^N}(0, \infty; \mathbb{R}^n)$. The cost of $A_i$ is given by

$$J_i(u) = \mathbb{E} \int_0^\infty \left\{ |x(t) - \Gamma x_i(t)|^2 + |u(t)|^2 \right\} dt.$$ 

(27)

In what follows, we study the following problem.

(P2). Seek a set of decentralized control laws to optimize the social cost $J_{soc}$ for the system (1), (27) where $J_{soc}(u) = \sum_{i=1}^N J_i(u)$.

We first introduce some definitions. Consider the system

$$dy = (Ay + Bu)dt + (Cy + Du)dW(t) + (C_0y + D_0u)dW_0(t), \quad y(0) = y,$$

(28)

$$z = Fy,$$

(29)

where $y(t) \in \mathbb{R}^n$, and $W(t), W_0(t)$ are 1-dimensional Brownian motions.

Definition 3.1 The system (28) with $u = 0$ (or simply $[A, C, C_0]$) is said to be mean-square stable, if for any initial $y$, there exists $c > 0$ such that

$$\mathbb{E} \int_0^\infty |y(t)|^2 dt \leq c.$$ 

Definition 3.2 The system (28) (or $[A, C, C_0; B; D, D_0]$) is said to be mean-square stabilizable, if there exists a control law $u^*(t) = Ky(t)$ such that for any initial $y$, the closed-loop system is mean-square stable.

$$dy(t) = (A + BK)y(t)dt + (C + DK)y(t)dW(t) + (C_0 + D_0K)y(t)dW_0(t).$$

Definition 3.3 ([51]) The system (28)-(29) (or simply $[A, C, C_0; F]$) is said to be exactly detectable, if there exists a $T_0 \geq 0$ such that for any $T > T_0$, $z(t) = 0, u(t) = 0, a.s.$, $0 \leq t \leq T$ implies

$$\lim_{t \to \infty} \mathbb{E}[y(t)^T y(t)] = 0.$$ 

We now introduce the following basic assumptions:

A5) The system $[A, C, C_0; B, D, D_0]$ is stabilizable, and $[A + G, C_0; B, D_0]$ is stabilizable.

A6) $Q \succeq 0, R > 0, [A, C, C_0; \sqrt{Q}]$ is exactly detectable, and $[A + G, C_0; \sqrt{Q}(I - \Gamma)]$ is exactly detectable.

Similar to Theorem 2.1, we have the following result.

Theorem 3.1 Assume $A, A_5)-A_6$ hold. Then, for a sufficiently large $N$, $(P2)$ has an optimal control $u \in \mathcal{U}_c$ such that $x_i, i \geq 1$ in $L^2_{\mathbb{R}^N}(0, \infty; \mathbb{R}^n)$ if and only if the following equation system admits a set of solutions $(x_i, \pi_i, \beta_i^j)_{j=0}^N, i \geq 1$ in $L^2_{\mathbb{R}^N}(0, \infty; \mathbb{R}^n)$:

$$\begin{aligned}
\dot{x}_i &= (A\dot{x}_i + B\dot{u}_i + G\dot{x}_i^N + f) dt + (C\dot{x}_i + D\dot{u}_i + \sigma) dW_i + (C_0\dot{x}_i + D_0\dot{u}_i + \sigma_0) dW_0, \quad \dot{x}_i(0) = x_{0i},
\pi_i &= -(A^T \pi_i + G^T p^N + C^T \beta_i^N) dt
+ (Q \dot{x}_i - Q_T \dot{x}_i^N - \eta) dt + \sum_{j=0}^N \beta_i^j dW_j,
\end{aligned} \quad j = 0, 1, \ldots, N$$

where the optimal control $\dot{u}_i (i = 1, \ldots, N)$ satisfies

$$\dot{R}_{\pi_i} + B^T \pi_i + D^T \beta_i^0 + D_0^T \beta_i^1 = 0.$$ 

Proof. (Sufficiency). The proof is similar to that of Theorem 2.1.

(Necessity). Since $\dot{x}_i \in L^2_{\mathbb{R}^N}(0, \infty; \mathbb{R}^n)$, we have

$$\mathbb{E} \int_0^\infty |\dot{x}_i(t)|^2 dt \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E} \int_0^\infty |\dot{x}_i(t)|^2 dt < \infty.$$ 

Based on a similar derivation as in Section 2.1, we only need to prove $p_i \in L^2_{\mathbb{R}^n}(0, \infty; \mathbb{R}^n)$. Let $p_i = P_N \dot{x}_i + K_N \dot{x}_i^N + \varphi_N$. Similarly, we have that $P_N, K_N, \varphi_N$ satisfy

$$A^T P_N + P_N A + C^T (P_N + K_N) C + Q + C_0^T P_N C_0 - \Psi_N T_N^{-1} \Psi_N = 0, \quad (30)$$

$$+ (A + G)^T K_N + K_N (A + G) + C_0^T K_N C_0 + C^T P_N + P_N G - (\Psi_N + \Theta_N) \Sigma (\gamma_N + D_0^T K_N D_0)^{-1} \Sigma (\Psi_N + \Theta_N) + \Psi_N \Sigma^{-1} \Psi_N - Q_T = 0, \quad (31)$$

$$d\varphi_N = - \left\{ [A + G - B (\Sigma_N + D_0^T K_N D_0)^{-1} \Sigma (\Psi_N + \Theta_N)]^T \nabla \varphi_N + [C - \Sigma_N (\gamma_N + D_0^T K_N D_0)^{-1} \Sigma (\Psi_N + \Theta_N) \Sigma \nabla \varphi_N + \nabla \varphi_N] \right\}.
$$

(32)

Let $\Pi_N = P_N + K_N$. Then, $\Pi_N$ satisfies

$$\begin{aligned}
(A^T + C^T (P_N + K_N) A + C_0^T (\Pi_N + K_N) C_0 + C^T P_N C + \Psi_N \Sigma^{-1} \Psi_N = 0. \quad (33)
\end{aligned}$$

From A5)-A6) and the continuous dependence of the solution on the parameter (see e.g. [11]), for a sufficiently large $N$, (30) and (33) admit solutions such that $[A + G, C_0; \sqrt{Q}(I - \Gamma)]$ is exactly detectable.
$G = B \hat{Y}_N^{-1}(\Psi_N + \Theta_N), C_0 - D_0 \hat{Y}_N^{-1}(\Psi_N + \Theta_N)]$ is stable. By [39, Lemma 2.5], (32) admits a unique solution $\phi_N \in \mathbb{L}_2^2(0, \infty; \mathbb{R}^n).$ Thus, we obtain $p_i \in \mathbb{L}_2^2(0, \infty; \mathbb{R}^n).$ \hfill \square

Based on the above discussion, we may construct the following decentralized control laws for Problem (P2):

\begin{equation}
\hat{u}_i = -\Psi^{-1}(x_i - \bar{x}) - \Psi^{-1}(\Psi + \Theta)\hat{x} + \psi, \quad t \geq 0, \quad i = 1, \cdots, N, \tag{34}
\end{equation}

where

$$\Psi = B^T P + D^T PC + D_0^T PC_0,$$ $\Theta = B^T (\Pi - P) + D_0^T (\Pi - P) C_0,$ and $\psi = D^T P \sigma + D_0^T \Pi \sigma_0 + B^T \varphi + D_0^T \gamma.$

In the above, $P$ and $\Pi$ satisfy

\begin{equation}
(A + G)^T \Pi + \Pi(A + G) - (\Psi + \Theta)^T \hat{Y}^{-1}(\Psi + \Theta) + C_0^T \Pi C_0 + C^T PC + Q - \Psi^T \Psi = 0, \tag{35}
\end{equation}

and $\varphi, \gamma, \bar{x} \in \mathbb{L}_2^2([0, \infty), \mathbb{R}^n)$ are determined by

\begin{equation}
d\varphi = -\left\{[A + G - B \hat{Y}^{-1}(\Psi + \Theta)]^T \varphi + \Pi f + [C_0 - D_0 \hat{Y}^{-1}(\Psi + \Theta)]^T \Pi \sigma + \gamma \right\} dt + \gamma dW_0, \tag{37}
\end{equation}

\begin{equation}
d\bar{x} = \left\{[A + G - B \hat{Y}^{-1}(\Psi + \Theta)] \bar{x} - B \hat{Y}^{-1} \psi + f\right\} dt + \left\{[C_0 - D_0 \hat{Y}^{-1}(\Psi + \Theta)] \bar{x} - D_0 \hat{Y}^{-1} \psi_0 + \sigma_0\right\} dW_0, \tag{38}
\end{equation}

where $\bar{x}(0) = \bar{x}_0.$

Here, (37) is a backward SDE and $(\varphi, \gamma)$ is its solution.

**Lemma 3.1** When (A2) and (A5)-(A6) holds, we have

(i) (35) admits a unique solution $P \geq 0$ such that $[A, C, C_0]$ is mean-square stable.

(ii) (36) admits a unique solution $\Pi \geq 0$ such that $[A + G - B \hat{Y}^{-1}(\Psi + \Theta), C_0 - D_0 \hat{Y}^{-1}(\Psi + \Theta)]$ is mean square stable;

(iii) (37)-(38) have a solution $\varphi, \bar{x} \in \mathbb{L}_2^2([0, \infty), \mathbb{R}^n).$

**Proof.** From (A5)-(A6), we obtain that (35) admits a unique solution $P \geq 0$ such that $[A, C, C_0]$ is mean-square stable (i.e., $P$ is a stabilizing solution; see e.g., [34], [51]). Note that $[A + G, C_0; (CT PC + Q - \Theta)]^{1/2}$ is exactly detectable. It follows that (36) admits a unique stabilizing solution $\Pi \geq 0.$ By [39, Lemma 2.5], (37) admits a unique solution $\varphi \in \mathbb{L}_2^2([0, \infty), \mathbb{R}^n).$ It is straightforward that $\bar{x} \in \mathbb{L}_2^2([0, \infty), \mathbb{R}^n).$ \hfill \square

For further analysis, we introduce the following assumption. We will show that this assumption is also necessary for the uniform stability of closed-loop systems.

**A7** $[\bar{A} + G, C_0]$ is mean-square stable, where $\bar{A}$ and $C_0$ are given by (18).

We now show that the closed-loop system under decentralized control (16) is uniformly mean-square stable.

**Theorem 3.2** Assume that (A2), (A5)-(A7) hold. Then, there exists an integer $N_0$ and a constant $c$ such that $N > N_0$ the following hold:

\begin{align}
\max_{1 \leq i \leq N} \mathbb{E} \int_0^\infty (|\dot{x}_i(t)|^2 + |\dot{\bar{x}}_i(t)|^2) dt < c, \tag{39}
\end{align}

\begin{align}
\mathbb{E} \int_0^\infty |\bar{x}_i(t) - \bar{x}(t)|^2 dt = O\left(\frac{1}{\sqrt{N}}\right). \tag{40}
\end{align}

**Proof.** See Appendix E. \hfill \square

**Remark 3.1** Due to the appearance of common noise, the mean-square approximation error between population state average $\bar{x}^{(N)}$ and aggregate effect $\bar{x}$ relies on the states of all agents while the second moment of the state $\dot{x}$ conversely depends on the approximation error $\dot{x}^{(N)} - \dot{x}.$ Thus, we need to analyze jointly $\bar{x}^{(N)} - \bar{x}$ and $\dot{x}_i, i = 1, \cdots, N.$ By tackling the relevant integral inequalities, we obtain the uniform stabilization of the systems and the consistency of MF approximation.

We now give two equivalent conditions for uniform stabilization of all the subsystems.

**Theorem 3.3** Let (A6) hold. Assume that (35)-(36) have symmetric solutions. Then, for (P2) the following statements are equivalent:

(i) For any initial condition $(\bar{x}_1(0), \cdots, \bar{x}_N(0))$ satisfying (A2), the following holds,

\begin{equation}
\sum_{i=1}^N \mathbb{E} \int_0^\infty (|\dot{x}_i(t)|^2 + |\dot{\bar{x}}_i(t)|^2) dt < \infty; \tag{41}
\end{equation}

(ii) (35) and (36) admit unique solutions such that $P \geq 0, \Pi \geq 0,$ and $[A + G, C_0]$ is stable;

(iii) (A5) and (A7) hold.

**Proof.** See Appendix E. \hfill \square

We now state the asymptotic optimality of the decentralized control.

**Theorem 3.4** Assume (A2), (A5)-(A7) hold. For Problem (P2), the set of decentralized strategies $\{\bar{u}_1, \cdots, \bar{u}_N\}$ given by (34) has asymptotic social optimality, i.e.,

\begin{equation}
\frac{1}{N} J_{soc}(\bar{u}) - \frac{1}{N} \inf_{u \in \mathbb{L}_2^2([0, \infty; \mathbb{R}^n])} J_{soc}(u) = O\left(\frac{1}{\sqrt{N}}\right).
\end{equation}

**Proof.** By a similar argument to the proof of Theorem 2.2 with Theorem 3.2, the conclusion follows. \hfill \square

**4 Numerical Example**

In this section, two examples are given to illustrate the effectiveness of the proposed decentralized control.
Example 4.1 Consider a scalar system with 30 agents in Problem (P2). For the system (1) and (27), take $A = -1, B = C = D = C_0 = D_0 = G = R = 0$ and $\eta(t) = \frac{1}{\sqrt{1 + t}}$. The initial states of 30 agents are taken independently from a normal distribution $N(10, 1)$. Note that $B = D = 0$. Both systems $[A, C, C_0; B, D, D_0]$ and $[A + G, C_0; B, D_0]$ are mean-square stabilizable independently from $\bar{\eta}$ essentially. Besides, it can be verified that $[\bar{\eta}] = [-0.5598, 0.4402]$ is mean-square stabilizable. We obtain that (A2), (A5)-(A7) hold. Under the control law (34), the state trajectories of agents are shown in Fig. 1. It appears that after the transient phase, all the states of agents reach the agreement and converge to 0 gradually.

![Fig. 1. Curves of 30 agents.](image)

The trajectories of $\bar{x}$ and $\bar{x}^{(N)}$ in (P2) are shown in Fig. 2. It can be seen that $\bar{x}$ and $\bar{x}^{(N)}$ coincide well, which illustrate the consistency of MF approximations.

![Fig. 2. Curves of $\bar{x}$ and $\bar{x}^{(N)}$.](image)

Example 4.2 We now consider the model in Example 4.1 where $A = 1, G = -0.8, R = -0.1$, and the other parameters are the same as above. In this case, we have $P = 0.5$ and $\Pi = -0.0648$. However, it can be verified that $[\bar{\eta}, C, \bar{\eta}] = [A + G - B^TW^{-1}(\Psi + \Theta), C_0 - D_0\bar{\eta}]$ and $[A + G, C_0]$ are mean-square stable. From Fig. 3, it can be seen that curves of $\bar{x}$ and $\bar{x}^{(N)}$ still coincide well.

5 Concluding Remarks

In this paper, we have considered uniform stabilization and social optimality for MF-LQ systems with common noise. By tackling coupled high-dimensional FBSDEs, we design the decentralized control laws for finite-horizon and infinite-horizon problems, respectively, and then, show their asymptotical social optimality. The necessary and sufficient conditions are given for uniform stabilization of all the subsystems.

An interesting generalization is to consider MF Stackelberg games with common noise. In this situation, the resulting FBSDEs are more complicated. How to obtain nonconservative conditions to ensure the solvability of FBSDEs is worthy of further study. Besides, it also deserves investigating further for applications in economics, smart grids and etc. [14], [25].

A Proof of Theorem 2.1

Proof. Suppose that $\bar{u}_i$ is a candidate of the optimal control of Problem (P1). Denote by $\dot{x}_i(\hat{\lambda})$ the state of agent $i$ under the control $\hat{\lambda}$. For any $v_i \in L^2(0, T; \mathbb{R}^r)$ and $\lambda \in \mathbb{R}(\lambda \neq 0)$, let $\bar{u}_i = \bar{u}_i + \lambda v_i$. Denote by $\dot{x}_i$ the solution of the following perturbed state equation

$$d\dot{x}_i = \left[A\dot{x}_i + B(\bar{u}_i + \lambda v_i) + \frac{G}{N} \sum_{i=1}^{N} \dot{x}_i + f \right] dt + \left[C\dot{x}_i + D(\bar{u}_i + \lambda v_i) + \sigma_0 \right] dW_0, \quad \dot{x}_i(0) = x_{i0}, \quad i = 1, 2, \cdots, N.$$

Let $y_i = (\dot{x}_i - \dot{x}_\lambda)/\lambda$. It can be verified that $y_i$ satisfies

$$dy_i = [Ay_i + Gy_i^{(N)} + Bv_i] dt + (Cy_i + Dv_i) dW_0 + (C_0y_i + D_0v_i) dW_0, \quad y_i(0) = 0.$$

Let $\{p_i, \beta_i \mid i = 1, \cdots, N, j = 0, 1, \cdots, N\}$ be a set of solutions to (3). Then, by Itô’s formula,

$$\sum_{i=1}^{N} \mathbb{E}[\langle p_i(T), y_i(T) \rangle] = \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ -\langle Q\dot{x}_i - Q_r\dot{x}_i^{(N)} - \eta_i \rangle, y_i + (B^T \beta_i + D^T \beta_i^{(N)} + D_0^T \beta_i^{(N)}) \right] dt. \quad (A.1)$$

From (2), we have

$$J_{soc}(\bar{u}) - J_{soc}(\hat{\lambda}) = 2\lambda \mathcal{I}_1 + \lambda^2 \mathcal{I}_2,$$
where \( \bar{u} = (\bar{u}_1, \cdots, \bar{u}_N) \), and
\[
\mathcal{I}_1 \triangleq \sum_{i=1}^N \mathbb{E} \left\{ \int_0^T \left[ \langle Q(\bar{x}_i - (\Gamma \bar{x}^{(N)}) + \eta) \rangle, y_i - \Gamma y^{(N)} \rangle \\
+ \langle R_i, v_i \rangle \right] dt + \langle H(\bar{x}_i(T) - (\Gamma_0 \bar{x}^{(N)}(T) + \eta_0)), y_i(T) - \Gamma_0 y^{(N)}(T) \rangle \right\},
\]
\[
\mathcal{I}_2 \triangleq \sum_{i=1}^N \mathbb{E} \left\{ \int_0^T \left( |y_i - \Gamma y^{(N)}|^2_Q + |v_i|^2_R \right) dt + |y_i(T) - \Gamma_0 y^{(N)}(T)|^2_H \right\}.
\]
A direct calculation with (A.1) gives
\[
\mathcal{I}_1 = \sum_{i=1}^N \mathbb{E} \int_0^T \langle \dot{R}_i, B^T p_i + D^T \beta_i^0 + D^T_0 \beta_i^0, v_i \rangle dt.
\]
By [30], [16], \( \mathcal{I}_2 \geq 0 \) equals that (P1) is convex. From a similar argument in [40], \( \bar{u} = (\bar{u}_1, \cdots, \bar{u}_N) \) is an optimal control of (P1) if and only if (P1) is convex and FBSDE (3) admits a solution \((\bar{x}_i, p_i, \beta_i^0, i, j = 1, \cdots, N)\). If Problem (P1) is uniformly convex, by [47], [38], (P1) admits a unique optimal control. \( \square \)

**B Proof of Theorem 2.2**

To prove Theorem 2.2, we need some lemmas. Consider the linear SDEs
\[
dz_i(t) = [A(t)z_i(t) + G(t)z^{(N)}(t) + b(t)]dt + [C(t)z_i(t) + \sigma(t)]dW_i(t) + [C_0(t)z_i(t) + \sigma_0(t)]dW_0(t), z_i(0) = z_{0i}, 1 \leq i \leq N. \tag{B.1}
\]

**Lemma B.1** Suppose that \( A, G, C \) and \( C_0 \) are bounded, and \( b, \sigma, \sigma_0 \in L^2_2(0, T; \mathbb{R}^n) \). Then, (B.1) admits a unique solution \( z_i \in L^2_2(0, T; \mathbb{R}^n) \). Particularly, there exists a constant \( c \), such that
\[
\sup_{0 \leq t \leq T} \sum_{i=1}^N \mathbb{E} \left[ |z_i(t)|^2 \right] \leq Nc \left[ \max_{1 \leq i \leq N} \mathbb{E} |z_{0i}|^2 \\
+ \mathbb{E} \int_0^T \left[ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2 \right] ds \right].
\]

**Proof.** By a similar argument in [47, Theorem 6.3], we obtain
\[
\mathbb{E} \left[ |z_i(t)|^2 \right] \leq c_0 \mathbb{E} \left[ |z_{0i}|^2 \right] + \int_0^t \mathbb{E} \left[ |z^{(N)}(s)|^2 \right] ds \\
+ \int_0^t \left[ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2 \right] ds \]
\[
\leq \frac{c}{N} \int_0^t \sum_{i=1}^N \mathbb{E} |z_i(s)|^2 ds + c_0 b(t),
\]
where
\[
h(t) \triangleq \max_{1 \leq i \leq N} \mathbb{E} |z_{0i}|^2 + \mathbb{E} \int_0^T \left[ |b(s)|^2 + |\sigma(s)|^2 + |\sigma_0(s)|^2 \right] ds.
\]
This further implies
\[
\sup_{0 \leq t \leq T} \sum_{i=1}^N \mathbb{E} |z_i(t)|^2 \leq c_0 b(t).
\]
By Gronwall’s inequality, we have \( \sum_{i=1}^N \mathbb{E} |z_i(t)|^2 \leq N c_0 e^{c_0 T} b(t) \). Let \( c = c_0 e^{c_0 T} \). The lemma follows. \( \square \)

**Lemma B.2** Let A1)-A4) hold. Under the control (16), the following holds:
\[
\sup_{0 \leq t \leq T} \sum_{i=1}^N \mathbb{E} |\bar{z}_i(t)|^2 \leq Nc.
\]

**Proof.** Since \( \psi, f, \sigma, \sigma_0 \in L^2_2(0, T; \mathbb{R}^n) \), by (15) and Lemma B.1, we have \( \sup_{0 \leq t \leq T} \sum_{i=1}^N \mathbb{E} |\bar{z}_i(t)|^2 \leq c \). From Lemma B.1 we obtain \( \sup_{0 \leq t \leq T} \sum_{i=1}^N \mathbb{E} |\bar{z}_i(t)|^2 \leq c \). \( \square \)

**Lemma B.3** Let A1)-A4) hold. Under the control (16), the following holds:
\[
\max_{0 \leq t \leq T} \mathbb{E} |\bar{x}^{(N)}(t) - \hat{x}(t)|^2 = O(1/N).
\]

**Proof.** Denote
\[
\bar{f} \triangleq b[T^{-1} \Psi - \hat{T}^{-1} (\Psi + \Theta)]\bar{x} - B\hat{T}^{-1} \psi + f, \\
\bar{\sigma} \triangleq D[T^{-1} \Psi - \hat{T}^{-1} (\Psi + \Theta)]\bar{x} - D\hat{T}^{-1} \psi + \sigma, \\
\bar{\sigma}_0 \triangleq D_0[T^{-1} \Psi - \hat{T}^{-1} (\Psi + \Theta)]\bar{x} - D_0 \hat{T}^{-1} \psi + \sigma_0.
\]
Let \( \z(t) = \hat{x}^{(N)}(t) - \hat{x}(t) \). It follows by (17) that
\[
d\hat{x}^{(N)}(t) = [(\hat{A}(t) + G(t))\hat{x}^{(N)}(t) + \bar{f}(t)]dt \\
+ \frac{1}{N} \sum_{i=1}^N [\hat{C}(t)\hat{x}_i(t) + \bar{\sigma}(t)dW_i(t) \\
+ [C_0(t)\hat{x}_i^{(N)}(t) + \bar{\sigma}_0(t)]dW_0(t).
\]
From this and (15), we have
\[
d\z(t) = [\hat{A}(t) + G(t)]\z(t)dt \\
+ \frac{1}{N} \sum_{i=1}^N [\hat{C}(t)\hat{x}_i(t) + \sigma(t)dW_i(t) \\
+ C_0(t)dW_0(t), \z(0) = \hat{x}^{(N)}(0) - \hat{x}_0.
\]
\]

\[10\]
By A2) and Lemma B.1, one can obtain

\[
\begin{align*}
\sup_{0 \leq t \leq T} E|\bar{\xi}^{(N)}(t) - \xi(t)|^2 & \leq \sup_{0 \leq t \leq T} E|\zeta(t)|^2 \\
& \leq cE\left\{ E|\bar{\xi}^{(N)}(0) - \bar{\xi}_0|^2 \\
& \quad + E\left[ \frac{1}{N} \sum_{i=1}^{N} \int_0^T |\tilde{C}(t)\tilde{x}_i(t) + \sigma(t)|^2 dt \right] \right\} \\
& \leq \frac{c}{N} \left\{ \max_{1 \leq i \leq N} E|\bar{x}_i(0)|^2 + \frac{1}{N} \sum_{i=1}^{N} E\int_0^T |\tilde{C}(t)\tilde{x}_i(t) + \sigma(t)|^2 dt \right\} \\
& \leq \frac{c}{N} \left\{ \max_{1 \leq i \leq N} E|\bar{x}_i(0)|^2 + \max_{1 \leq i \leq N} E\int_0^T |\tilde{C}(t)\tilde{x}_i(t) + \sigma(t)|^2 dt \right\}.
\end{align*}
\]

By Lemma B.2, the proof is completed. \( \square \)

**Proof of Theorem 2.2.** We first prove that for \( u \in U_c \), \( J^F_{\text{loc}}(u) < \infty \) implies \( J^F_{\text{loc}}(u) < \infty \) for all \( i = 1, \ldots, N \). By A3) and [40, Proposition 3.1], we have

\[ \delta_0 \sum_{i=1}^{N} E \int_0^T |u_i(t)|^2 dt - c \leq J^F_{\text{loc}}(u) < \infty, \]

which implies \( \sum_{i=1}^{N} E \int_0^T |u_i(t)|^2 dt < c_1 \). By Lemma B.1 and (1), we have

\[ \sum_{i=1}^{N} E \int_0^T |x_i(t)|^2 dt \leq Nc. \tag{B.2} \]

Denote \( \tilde{x} = x - \hat{x}, \tilde{u} = u - \hat{u} \) and \( \tilde{x}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i \). Then,

\[ E \int_0^T (|\tilde{x}_i|^2 + |\tilde{u}_i|^2) dt < \infty. \tag{B.3} \]

It follows from (1) and (17) that

\[
\begin{align*}
d\tilde{x}_i = & \left( A\tilde{x}_i + C\tilde{x}^{(N)} + B\tilde{u}_i \right) dt + \left( C\tilde{x}_i + D\tilde{u}_i \right) dW_i \\
& + \left( C_0\tilde{x}_i + D_0\tilde{u}_i \right) dW_0, \quad \tilde{x}_i(0) = 0.
\end{align*}
\]

We have \( J^F_{\text{loc}}(u) = \sum_{i=1}^{N} (J^F_{\text{i}}(\tilde{u}) + J^F_{\text{I}}(\tilde{u}) + I_i) \), where

\[
J^F_{\text{i}}(\tilde{u}) \triangleq E \int_0^T \left( |\tilde{x}_i - \Gamma\tilde{x}^{(N)}(T)|^2 + |\tilde{u}_i|^2 \right) dt + E[\hat{x}_i(T) - \Gamma_0\hat{x}^{(N)}(T)]^2 dt,
\]

\[
I_i \triangleq 2E \int_0^T \left( \left( \tilde{x}_i - \Gamma\tilde{x}^{(N)} - \eta \right)^T Q \left( \tilde{x}_i - \Gamma\tilde{x}^{(N)} \right) + \tilde{u}_i^T R\tilde{u}_i \right) dt \\
+ 2E \left( H(\tilde{x}_i(T) - \Gamma_0\hat{x}^{(N)}(T) + \eta_0) \right)^T Q(\tilde{x}_i - \Gamma\tilde{x}^{(N)} + \eta_0) dt + E[\hat{x}_i(T) - \Gamma_0\hat{x}^{(N)}(T) + \eta_0) ]^T H(\tilde{x}_i(T) - \Gamma_0\hat{x}^{(N)}(T)) dt.
\]

By A3) together with [30, Lemma 1], \( \sum_{i=1}^{N} \bar{J}^F_{\text{loc}}(\tilde{u}) \geq 0 \). We only need to prove \( \frac{1}{N} \sum_{i=1}^{N} I_i = O(1/\sqrt{N}) \). Note that

\[
\begin{align*}
& \sum_{i=1}^{N} I_i = \sum_{i=1}^{N} 2E \int_0^T \left( \tilde{x}_i^T (Q\tilde{x}_i - Q_T\tilde{x} - \eta) + \tilde{u}_i^T R\tilde{u}_i \right) dt \\
& \quad - \sum_{i=1}^{N} 2E \int_0^T \left( \tilde{x}_i^T (\tilde{x}^{(N)} - \tilde{x})^T Q_T\tilde{x}_i dt + E[\tilde{x}_i^T (T) H_{\gamma_0} (\tilde{x}^{(N)}(T) - \tilde{x}(T))] \right) \\
& \quad + 2N E \left( T \left( H\tilde{x}_i(T) - H_{\gamma_0}\tilde{x}_i(T) - \eta_0 \right) \right). \tag{B.5}
\end{align*}
\]

Let \( \tilde{H}_i = P\tilde{x}_i + K\tilde{x}_i \). By (8)-(10) and Itô’s formula, we obtain

\[
\begin{align*}
d\tilde{H}_i = & \left[ A^T P\tilde{x}_i + G^T P\tilde{x}^{(N)} + C_0^T P\tilde{u}_i + C^T P\tilde{u}_i \right] dt + \left( C_0 P\tilde{x}_i + D_0 P\tilde{u}_i \right) dW_i \\
& + \left( C P\tilde{x}_i + D P\tilde{u}_i \right) dW_0, \quad \tilde{H}_i(0) = 0.
\end{align*}
\]

Note that by (16), \( R\tilde{u}_i = - (B\tilde{P}_i + D\tilde{B}_i + D\tilde{B}_i) \). From (B.4) and (B.6), we obtain

\[
\begin{align*}
& \sum_{i=1}^{N} E \int_0^T \left( \tilde{H}_i(T) \right) dt \\
& \quad = \sum_{i=1}^{N} E \left\{ \tilde{H}_i \left( T \left( \tilde{x}^{(N)}(T) - \tilde{x}(T) \right) \right) \right\} dt \\
& \quad + \sum_{i=1}^{N} E \left\{ \tilde{H}_i \left( \tilde{x}^{(N)}(T) - \tilde{x}(T) \right) \right\} dt
\end{align*}
\]

From this and (B.5), we obtain

\[
\begin{align*}
& \sum_{i=1}^{N} I_i = 2E \int_0^T \left( \tilde{x}^{(N)}(T) - \tilde{x}(T) \right) \left( G^T P + C^T P \right) \tilde{x}^{(N)}(T) dt \\
& \quad + 2E \left( \left( \tilde{x}^{(N)}(T) - \tilde{x}(T) \right)^T \right) H_{\gamma_0} \tilde{x}^{(N)}(T) dt.
\end{align*}
\]

By Lemma B.3, (B.2) and (B.3), we obtain

\[
\begin{align*}
& \frac{1}{N} \sum_{i=1}^{N} I_i \leq cE \int_0^T \left| \tilde{x}^{(N)}(T) - \tilde{x}(T) \right|^2 dt \cdot E \int_0^T \left| \tilde{x}^{(N)}(T) \right|^2 dt + cE \left( \tilde{x}^{(N)}(T) - \tilde{x}(T) \right)^2 \cdot E \left| \tilde{x}^{(N)}(T) \right|^2,
\end{align*}
\]

which implies \( \frac{1}{N} \sum_{i=1}^{N} I_i = O(1/\sqrt{N}) \). \( \square \)
Proof of Theorem 2.3

To prove Theorem 2.3, we need two lemmas.

Denote $\mathbb{E}_0(\cdot) \triangleq \mathbb{E}(\cdot|\mathcal{F}_0)$. Consider the (conditional) MF type system

$$dz_i = (Az_i + Bu_i + GE_{F_0}(z_i) + f)dt + (Cz_i + Du_i + \sigma)dw_i,$$

with the cost functional

$$J_i(u_i) = \mathbb{E} \int_0^T (z_i - \Gamma E_{F_0}(z_i) - \eta)^2 + |u_i|_Q^2 dt + \mathbb{E}|z_i(T) - \Gamma 0E_{F_0}(z_i(T)) - \eta_0|^2 dt.$$  \hspace{1cm} (C.2)

The admissible control set is given by

$$U_i = \{u_i \mid u_i(t) \text{ is adapted to } \sigma(z_i(0), W_i(0), W_0(s)), 0 \leq s \leq t, \mathbb{E} \int_0^T |u_i(t)|^2 dt < \infty \}.  \hspace{1cm}$$

Lemma C.1 Assume A1-A4 hold. For the system (C.1)-(C.2), the optimal control is given by

$$\hat{u}_i = -Y^{-1}\Psi(z_i - E_{F_0}(z_i)) - \hat{Y}^{-1} [\Psi + \Theta] E_{F_0}(z_i) + \psi,$$

and the optimal cost is

$$\inf_{u_i \in U_i} J_i(u_i) = \mathbb{E} \left[ (x_{i0} - \bar{x}_0)^T P (x_{i0} - \bar{x}_0) + \bar{x}_0^T \Pi \bar{x}_0 + 2\phi^T(0)\bar{x}_0 \right] + C T.$$ \hspace{1cm} (C.3)

Proof. The proof is similar to Theorem 2.6 of [13]. \hspace{1cm} \Box

After applying the control (C.3) into (C.1), we have

$$d\hat{z}_i = [A \hat{z}_i + G E_{F_0}(\hat{z}_i) + B(Y^{-1}\Psi - \hat{Y}^{-1}(\Psi + \Theta))E_{F_0}(\hat{z}_i) - B Y^{-1}\psi + f]dt + \{C \hat{z}_i + D(Y^{-1}\Psi - \hat{Y}^{-1}(\Psi + \Theta))\}dw_i + \{\hat{C}_0 \hat{z}_i + D_0(Y^{-1}\Psi - \hat{Y}^{-1}(\Psi + \Theta))E_{F_0}(\hat{z}_i) - D_0 \hat{Y}^{-1}\psi + \sigma\}dw_i.$$ \hspace{1cm} (C.4)

Lemma C.2 Assume that A1-A4 hold. Then, $\mathbb{E} \int_0^T |\hat{x}_i - \hat{z}_i|^2 dt = O\left(\frac{1}{N}\right)$. \hspace{1cm} \Box

Proof. From (C.4),

$$dE_{F_0}(\hat{z}_i) = [A + G - B \hat{Y}^{-1}(\Psi + \Theta)]E_{F_0}(\hat{z}_i) - B \hat{Y}^{-1}\psi + f]dt + \{C_0 - D_0 \hat{Y}^{-1}(\Psi + \Theta)\}E_{F_0}(\hat{z}_i) - D_0 \hat{Y}^{-1}\psi + \sigma\}dw_i.$$ \hspace{1cm}

By comparing (C.4) with (15), we can verify that $E_{F_0}(\hat{z}_i) = \bar{x}_0$. From (17),

$$d(\hat{x}_i - \hat{z}_i) = A(\hat{x}_i - \hat{z}_i)dt + G(\hat{x}(N) - E_{F_0}(\hat{z}_i))dt + C(\hat{x}_i - \hat{z}_i)dw_i + \check{C}(\hat{x}_i - \hat{z}_i)dw_i.$$ \hspace{1cm}

This gives

$$\hat{x}_i(t) - \hat{z}_i(t) = \int_0^T \Phi_i(t - \tau) \mathbb{E}(\hat{x}(N)(\tau) - E_{F_0}(\hat{z}_i)(\tau))d\tau,$$

where $\Phi_i$ satisfies

$$d\Phi_i(t) = A(t)\Phi_i(t)dt + C(t)\Phi_i(t)dw_i + \check{C}(t)\Phi_i(t)dw_i, \Phi_i(0) = I.$$ \hspace{1cm}

By Schwarz’s inequality and Lemma B.3,

$$\frac{1}{N} J_{soc}(\bar{u}) = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \int_0^T \left[ |\hat{x}_i - (\Gamma \hat{x}(N) + \eta)|_Q^2dt + |Y^{-1}\Psi(\hat{x}_i - \bar{x}) + \hat{Y}^{-1}(\Psi + \Theta)\hat{x}_i|_R^2 dt \right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \int_0^T \left[ |\hat{z}_i - \Gamma E_{F_0}(\hat{z}_i) - \bar{x} + \hat{x}_i - \bar{z}_i|_Q^2 + |\hat{x}_i - \bar{z}_i + \bar{z}_i|_Q + \hat{Y}^{-1}(\Psi + \Theta)|\hat{z}_i - \bar{z}_i|_Q^2 + \hat{Y}^{-1}(\Psi + \Theta)|\hat{z}_i - \bar{z}_i + \bar{z}_i - \hat{E}_{F_0}(\hat{z}_i)|_Q^2 \right] dt.$$ \hspace{1cm}

Note that $J_{soc}(\bar{u}) = \sum_{i=1}^N J_i(\bar{u}) = N J_i(\bar{u})$. By Schwarz’s inequality, and Lemma C.2,

$$\left|\frac{1}{N} J_{soc}(\bar{u}) - \frac{1}{N} J_{soc}(\bar{u}) \right| = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \int_0^T \left[ |\hat{x}_i - \bar{z}_i|_Q^2 + |\mathbb{E}[\hat{x}(N) - E_{F_0}(\hat{z}_i)]|_Q^2 dt + C_2 \frac{N}{2} \sum_{i=1}^N \left( \mathbb{E} \int_0^T |\hat{x}_i - \bar{z}_i|_Q^2 dt \right)^{1/2} \right] + \frac{C_2}{2} \sum_{i=1}^N \left( \frac{N}{2} \mathbb{E} \int_0^T |\mathbb{E}[\hat{x}(N) - E_{F_0}(\hat{z}_i)]|_Q^2 dt \right)^{1/2} = O(1/\sqrt{N}).$$ \hspace{1cm}

Letting $N \to \infty$, by Lemma C.1, the theorem follows. \hspace{1cm} \Box
D Proofs of Proposition 2.2 and Theorem 2.4

Proof of Proposition 2.2. Denote by $x_i^*$ the corresponding state under the control $u_i^*$, $i = 1, \ldots, N$. Let $\delta u_i = u_i - u_i^*$, $\delta x_i = x_i - x_i^*$ and $\delta x_i^{(N)} = \sum_{i=1}^{N} \delta x_i$. Assume $u_i^*, u_i \in U_i^0$. Denote by $\delta J_{soc}^F$ the variation of $J_{soc}^F$ with $\delta u_i$. Then, we have

$$
\delta J_{soc}^F = \sum_{i=1}^{N} E \left\{ \int_0^T \left[ \langle Q(x_i^* - (\Gamma x_i^{(N)} + \eta)) \rangle, \delta x_i - \delta x_i^{(N)} \rangle + \langle Ra_i^*, \delta u_i \rangle \right] dt + \langle H(x_i^*(T) - (\Gamma_0 x_i^{(N)}(T) + \eta_0)), \delta x_i(T) - \Gamma_0 \delta x_i^{(N)}(T) \rangle \right\}.
$$

Let $\{p_i^*, \beta_i^{0*}, i = 1, \cdots, N, j = 0, 1, \cdots, N\}$ be a solution to the second equation in (19). By Itô's formula,

$$
\sum_{i=1}^{N} \delta J_{soc}^F = \sum_{i=1}^{N} E \left\{ \int_0^T \left[ \langle -[Qx_i^* - Q_{x_i} x_i^{(N)} - \eta + G T p_i^*], \delta x_i \rangle + \langle B T p_i^* + DT \beta_i^{0*} + DT \beta_i^{1*}, \delta u_i \rangle \right] dt. \right\}
$$

From (D.1) and (D.2), we obtain

$$
\frac{\delta J_{soc}^F}{\delta u_i} = \frac{\sum_{i=1}^{N} \int_0^T \langle Ra_i^* + BT p_i^* + DT \beta_i^{0*} + DT \beta_i^{1*}, \delta u_i \rangle dt.}{\sum_{i=1}^{N} \int_0^T \langle Ra_i^* + BT p_i^* + DT \beta_i^{0*} + DT \beta_i^{1*}, \delta u_i \rangle dt.}
$$

Note that $u_i^*, \delta u_i \in H_i^1$. By the smoothing property of conditional mathematical expectation,

$$
\delta J_{soc}^F = \sum_{i=1}^{N} E \left\{ \int_0^T \langle Ra_i^* + BT p_i^* + DT \beta_i^{0*} + DT \beta_i^{1*}, \delta u_i \rangle dt. \right\}
$$

From (17) and (38), we have

$$
d(\zeta(t)) = (\bar{A} \zeta(t) + f) dt + (\bar{C} \zeta(t) + \bar{\sigma}) dW(t) + \frac{1}{N} \sum_{i=1}^{N} (\bar{C} \zeta(t) + \bar{\sigma}) dW_i(t)
$$

From (A5)-A6), there exists a constant $c$ such that $P > 0$. By Itô's formula, we have

$$
E[\zeta(T) P \zeta(T) - \zeta(0) P \zeta(0)]
$$

which gives

$$
E \left\{ \int_0^T \langle \bar{C} \zeta(t) + \bar{\sigma} \rangle d\bar{P} \zeta(T) dt \right\}
$$

Let $P$ satisfy

$$
PA + A^T P + CT PC + C^T_0 PC_0 = -2I.
$$
From A5)-A6), we have $P > 0$. By Itô's formula, we obtain
\[
\mathbb{E}[\dot{x}_i^T(T)P\dot{x}_i(T) - \dot{x}_i^T(0)P\dot{x}_i(0)] \\
\leq \mathbb{E} \left[ \int_0^T \left( \dot{x}_i^T(P\ddot{x} + AT^T + C^T P\dot{C} + \dot{C}^T P\ddot{C})\dot{x}_i + \dot{x}_i^T(\dot{P}G + G^T\dot{P})\dot{x}_i + \ddot{x}_i^T(\dot{P}f + C^T\dot{P} + \dot{C}^T P\dot{P})\dot{x}_i \right) dt \right]
\leq -\mathbb{E} \left[ \int_0^T (\dot{x}_i^T\dot{x}_i) dt \right] + \alpha_T, \text{ a.s.},
\]
where
\[
\alpha_T = \mathbb{E} \left[ \int_0^T \left( \dot{x}_i^T(\ddot{x}_i + \dddot{x}_i(N)) + \dot{\sigma}_T\dot{x}_i \right) dt \right] + \left| \int_0^T (\dot{P}f + C^T\dot{P} + \dot{C}^T P\dot{P})\dot{x}_i dt \right|.
\]
This implies \( \mathbb{E} \left[ \int_0^T |\dot{x}_i(t)|^2 dt \right] \leq \mathbb{E}[\dot{x}_i^T(0)\dot{x}_i(0)] + \alpha_T, \) which with (E.3) further gives
\[
\mathbb{E} \int_0^T |\dot{x}_i(t)|^2 dt \leq c_0 \mathbb{E} \int_0^T |\dot{x}_i(N)(t)|^2 dt + c_3 \\
\leq 2c_0 \mathbb{E} \int_0^T (|\dot{x}_i(t)|^2 + |\zeta(t)|^2) dt + c_3 \\
\leq 2c_3 \mathbb{E} \left[ \int_0^T |\dot{x}_i(t)|^2 dt + \frac{c_0}{N} \max_{1 \leq i \leq N} \mathbb{E} \int_0^T |\dot{x}_i(t)|^2 dt \right] + c_4.
\]
Thus, there exists an integer $N_0$ such that $N > N_0$, the following holds:
\[
\max_{1 \leq i \leq N} \mathbb{E} \int_0^T |\dot{x}_i(t)|^2 dt \leq c_0 \mathbb{E} \int_0^T |\dot{x}(t)|^2 dt + c_6.
\]
Note $\dot{x} \in L_2^2([0, \infty), \mathbb{R}^n)$. It follows that
\[
\max_{1 \leq i \leq N} \mathbb{E} \int_0^T |\dot{x}_i(t)|^2 dt \leq c.
\]
This together with (E.3) gives (40).

**Proof of Theorem 3.3.** (iii)⇒(i) has been given in Theorem 3.2. We now show (i)⇒(ii). By (E.1),
\[
\frac{d\mathbb{E}[\varphi(\dot{x}_i)]}{dt} = \left[ A \mathbb{E}[\varphi(\dot{x}_i)] + \mathbb{E}\varphi(\dot{x}_i) \right] + \int_0^T \Phi_i^{-1}(\tau) [G\dot{x}_i(N) + f - \dot{C}\sigma - C_0\sigma_0] dW_0, \quad \mathbb{E}[\varphi(\dot{x}_i(0))] = \bar{x}_0.
\]
It follows from A2) that
\[
\mathbb{E}[\varphi(\dot{x}_i)] = \mathbb{E}[\varphi(\dot{x}_j)] = \mathbb{E}[\varphi(\dot{x}_i(N))], \quad j \neq i.
\]
By comparing (E.4) with (38), we obtain $\mathbb{E}[\varphi(\dot{x}_i)] = \bar{x}$. Note that
\[
\mathbb{E}[\varphi]^2 = \mathbb{E}[\varphi]^2 = \mathbb{E}\left\{ \mathbb{E}[\varphi]^2 \right\} \leq \mathbb{E}\left\{ \mathbb{E}[\varphi]^2 \right\} = \mathbb{E}[\varphi]^2.
\]
It follows from (41) that
\[
\mathbb{E} \int_0^\infty |\dot{x}(t)|^2 dt < \infty.
\]
By (38), we have
\[
\dot{x}(t) = \Phi_0(t) \left[ \dot{x}_0 + \int_0^t \Phi_0^{-1}(\tau) h(\tau) d\tau \right.
\]
\[
+ \left. \int_0^t \Phi_0^{-1}(\tau)(\sigma(\tau) - D_0 \dot{T}^{-1}\psi(\tau)) dW_0(\tau) \right],
\]
where $h = f - B\dot{T}^{-1}\psi - C_0(\sigma - D_0 \dot{T}^{-1}\psi)$, and $\Phi_0$ satisfies
\[
d\Phi_0 = (A + G - B\dot{T}^{-1}(\Psi + \Theta))\Phi_0 dt
\]
\[
+ (C_0 - D_0 \dot{T}^{-1}(\Psi + \Theta))\Phi_0 dW_0, \quad \Phi_0(0) = I.
\]
By the arbitrariness of $\dot{x}_0$ and (E.5), we obtain that $[A + G; C_0; D_0]$ is stabilizable. Note that $\mathbb{E}[\varphi(\dot{x}(N))]^2 \leq \frac{1}{N} \sum_{i=1}^N \mathbb{E}[\varphi]^2$. Then, from (41) we have
\[
\mathbb{E} \int_0^\infty |\dot{x}(t)|^2 dt < \infty.
\]
This leads to $\mathbb{E} \int_0^\infty |G\dot{x}(N) + f|^2 dt < \infty$. By (E.1), we obtain
\[
\mathbb{E}[\varphi]^2 = \mathbb{E} \left[ \int_0^T \Phi_i^{-1}(\tau) [G\dot{x}_i(N) + f - \dot{C}\sigma - C_0\sigma_0] dW_0 + \int_0^T \Phi_i^{-1}(\tau) \sigma_0 dW_0(\tau) \right]^2,
\]
where $\Phi_i$ satisfies
\[
d\Phi_i = A\Phi_i dt + \dot{C}\Phi_i dW_0 + \bar{C}\Phi_i dW_0, \quad \Phi_i(0) = I.
\]
By (41) and the arbitrariness of $\dot{x}_0$, we obtain that $\mathbb{E} \int_0^\infty |\dot{x}(t)|^2 dt < \infty$, i.e., $[A, C, C_0; B, D, D_0]$ is stabilizable. On the other hand, from (E.5) and (E.6),
\[
\mathbb{E} \int_0^\infty |\varphi(t)|^2 dt = \mathbb{E} \int_0^\infty \left| \dot{x}(t) - \bar{x}(t) \right|^2 dt < \infty.
\]
Noting that $\varphi(t)$ satisfies (E.2), we obtain that $\mathbb{A} \mathbb{T}$ holds. (ii)⇒(iii). By a similar argument in [50], [51], [40], we can show $[A, C, C_0; B, D, D_0]$ is stabilizable. Since
Π ≥ 0, there exists an orthogonal $U$ such that $U^TΠU = \begin{bmatrix} 0 & 0 \\ 0 & Π_2 \end{bmatrix}$, where $Π_2 > 0$. From (36),

$$
(U^TΔU)^TΠU + U^TΠU(U^TΔU) + (U^TΔU)^TΠU(U^TΔU) + U^TΞU = 0,
$$

where $Δ = A + G - BA, Ξ_0 = C_0 - D_0Λ$ with $Σ \triangleq Σ^{-1}(Ψ + Φ)$, and

$$Ξ = C^TPC + Q - QT + Λ^T(R + D^TPD)Λ - Λ^TD^TPC - (D^TPC)^TΛ.$$

Let $Σ = R + D^TPD$. Then, $Ξ$ can be deformed into

$$Ξ = \Lambda - D^TPC \hat{Σ} \Lambda,$$

$$= \Lambda^T \hat{Σ}^T - D^TPC \hat{Σ} \Lambda + C^TPD^TPD + C^TPC + Q - QT = 0.$$

Note that

$$
\begin{bmatrix}
P & PD \\
D^TPR + D^TPD & P
\end{bmatrix}
\geq
\begin{bmatrix}
I & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
P & 0 \\
0 & R
\end{bmatrix}
\begin{bmatrix}
I & D
\end{bmatrix}.
$$

Thus, we have $\begin{bmatrix}
P & PD \\
D^TPR + D^TPD & P
\end{bmatrix} \geq 0$. By Schur’s lemma [34], $P - PD\hat{Σ}^T - D^TP\hat{Σ} \geq 0$. This gives $C^T(P - PD\hat{Σ}^T - D^TP\hat{Σ})C \geq 0$. From (E.8), $Ξ \geq 0$. By [51, Theorem 3.1] and $A_6), we obtain that $(Δ, Ξ_0; Σ^{1/2})$ is exactly detectable. Denote

$$U^TΔU = \begin{bmatrix}
Δ_{11} & Δ_{12} \\
Δ_{21} & Δ_{22}
\end{bmatrix},
\quad U^TΞU = \begin{bmatrix}
Ξ_{11} & Ξ_{12} \\
Ξ_{21} & Ξ_{22}
\end{bmatrix},
\quad U^TΞU = \begin{bmatrix}
Ξ_{11} & Ξ_{12} \\
Ξ_{21} & Ξ_{22}
\end{bmatrix}.$$

By pre- and post-multiplying by $ξ^T$ and $ξ$ where $ξ = [ξ_1^T, 0]^T$, it follows that $0 = ξ^TΣUξ$. From the arbitrariness of $ξ_1$, we obtain $Ξ_{11} = 0$. Since $Ξ$ is positive semi-definite, $Ξ_{22} = Ξ_{21} = 0$, and $Ξ_{22} ≥ 0$. By comparing each block matrix of both sides of (E.7), we obtain $Δ_{22} = Ξ_{22} = 0$. It follows from (E.7) that

$$Π_2Δ_{22} + Π_2Δ_{21} = (Δ_{22}Σ_{22} + Ξ_{22} = 0).$$

Let $ν = [ν_1^T, ν_2^T]^T = U^T\tilde{y}$, where $\tilde{y}^* = Κ\tilde{y}^*$. Then, we have

$$dν_1 = (Δ_{11}ν_1 + Δ_{12}ν_2)dt + (Δ_{11}ν_1 + Δ_{12}ν_2)dW_0,
\quad dν_2 = Δ_{22}ν_2dt + Δ_{22}ν_2dW_0.$$
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