A Pole Placement Based Output Tracking Control Scheme by Finite-and-Quantized Output Feedback

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Abstract—This paper proposes a finite-and-quantized output feedback output tracking control scheme for possibly non-minimum phase discrete-time linear systems that are subject to output quantization and saturation. An analytical pole placement-based control law is proposed by solely utilizing the finite-and-quantized output and the external reference output. The closed-loop stability and output tracking analysis are essentially different from the classical pole placement method. It needs to overcome some new technical issues caused by finite-and-quantized output feedback, such as how to realize closed-loop stability while restricting the finite quantization of the output measurement. This paper demonstrates that by appropriately designing the quantizer’s sensitivity, the proposed control law ensures all closed-loop signals are bounded, and the output tracking error converges to a certain residual set of the origin within a certain finite time, regardless of the existence of unstable zeros and poles in the control systems. Particularly, the residual set can be arbitrarily small under a specified design condition. Finally, a representative example validates the proposed control scheme.

Index Terms—Discrete-time, finite time, output tracking, pole placement control, quantized-output feedback

I. INTRODUCTION

DURING the past two decades, the problems of control systems subject to quantized and/or saturated constraints have attracted significant attention in the control community. Generally, classic state and output feedback control methods cannot be directly applied to control systems subject to quantization and saturation measurements, especially for the finite quantized case. Additionally, practical control systems suffer from measurement errors due to the sensor’s limitations in accuracy and magnitude. Moreover, compared to exact feedback control designs, the finite-and-quantized feedback control technique is more robust in anti-disturbances. Therefore, it is of theoretical and practical significance to develop new methods that improve the system performance for control systems subject to quantization and saturation problems.

To date, tremendous development has been made in quantized feedback control theory and applications. In the 1960s, [1] introduced the quantized control idea to reduce the computational burden in optimum designs. Since then, various quantized control methods have been developed, such as [2]–[6]. The quantizers in quantized feedback control systems are generally divided into static and dynamic, with the former having fixed quantized levels and quantized errors. Nevertheless, solely relying on static quantization is challenging to afford global or semi-global convergence. Therefore, [7] developed a dynamic quantized method, which, compared to the static, has an adjustable parameter called “the sensitivity” of the quantizer. Both quantizer types have been widely used in various control system classes, e.g., deterministic control systems [8], stochastic control systems [9], multi-agent consensus or formation [10], [11], and network control systems [12], [13]. In particular, the stabilization and tracking control problems have been extensively studied utilizing quantized control [14]–[16]. Furthermore, the saturation problem has also been considered in quantized control systems, where [17] studied the control problem of the systems subject to saturation and proposed several constructive anti-windup methods, and [18] investigated the robust stabilization problem of uncertain linear systems under saturated state feedback.

Recently, [19] established a basic framework for tracking control of discrete-time linear time-invariant (LTI) systems by using finite-and-quantized output feedback. Moreover, a new model reference control (MRC) scheme was developed by utilizing finite-and-quantized output feedback. However, such a control scheme is only suitable for controlling minimum-phase systems, still suffering from the fundamental control problem: how to effectively control a general class of discrete-time LTI systems covering the minimum and non-minimum phase cases. In [20] and [21], the authors demonstrated that the pole placement control (PPC) method achieves an appealing output tracking via the exact state or output feedback. However, it is an open research case whether a finite-and-quantized output feedback version of the classic PPC scheme is still valid. To our knowledge, such a problem has not been addressed yet. Compared with the classic PPC method, the finite-and-quantized feedback case suffers from some new technical problems, such as realizing closed-loop stability and achieving output tracking while restricting the finite quantization of the
output measurement. Hence, this paper aims to address these concerns systematically. The main contributions of this paper are as follows:

(i) Providing an affirmative "yes" answer to the problem of whether a finite-and-quantized output feedback version of the classic PPC law is still valid. In particular, a quantized-and-saturated output feedback version of the classic PPC law is analytically constructed.

(ii) Compared with the existing literature, the proposed control method has distinctive characteristics. First, it ensures that the output tracking error converges to a residual set of the origin within a certain finite time. Second, global convergence of the output tracking is achieved in the sense that the proposed control law is independent of the system’s initial conditions. Finally, the controlled plant is allowed to have unstable poles and zeros, i.e., the proposed control method is effective for both minimum-phase and non-minimum phase systems.

The remainder of this paper is as follows. Section II introduces the system model and control problems to be addressed. Section III reviews the fundamental PPC law and presents the details of the finite-and-quantized output feedback PPC scheme. Section IV presents some simulation examples, and finally, Section V concludes this work.

II. Problem Statement

This section presents the system model and the problems investigated in this paper.

System model. Consider the following discrete-time single-input and single-output (SISO) LTI system:

\[ A(z)y(t) = B(z)u(t), \ t \geq t_0, \]  

where \( t_0 \) is the initial moment of the system operation and \( A(z), B(z) \) are polynomials with constant coefficients of degree \( n \) and \( n - 1 \), respectively, i.e.,

\[ A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \]
\[ B(z) = b_{n-2}z^{n-2} + \cdots + b_1 z + b_0. \]

In this paper, \( z \) and \( z^{-1} \) denote the forward and backward shift operators, i.e., \( z[x](t) = x(t+1) \) and \( z^{-1}[x](t) = x(t-1) \), where \( t \in \{0, 1, 2, 3, \ldots\} \). \( x(t) \triangleq x(T) \) for a sampling period \( T > 0 \) and \( x(t) \) denotes any signal of any finite dimension. For the system model (1), \( y(t) \) cannot be measured accurately and one can only acquire its finite and quantized values denoted as \( q(y(t), \Delta(t)) \), where \( q \) is the quantizer to be given.

Dynamic quantizer. Let \( X(t) \in \mathbb{R} \) be any signal on \( \mathbb{R} \). Then, this paper’s quantizer is specified as

\[ q(X(t), \Delta(t)) = \begin{cases} 
M, & \text{if } X(t) > (M + \frac{1}{2})\Delta(t), \\
\left\lfloor \frac{X(t)}{\Delta(t)} \right\rfloor + \frac{1}{2}, & \text{if } -(M + \frac{1}{2})\Delta(t) < X(t) \leq (M + \frac{1}{2})\Delta(t), \\
M, & \text{if } X(t) \leq -(M + \frac{1}{2})\Delta(t), 
\end{cases} \]

where \( \Delta(t) \neq 0 \) depends on \( t \) and is called the sensitivity of \( q \). \( M \) is a positive integer, and \( |X(t)| \triangleq \max\{|k \in \mathbb{Z} : k < X(t)|\} \). Although the proposed method does not have any restrictive conditions on the used quantizer, this paper relies on the quantizer proposed in [7]. In this sense, the suggested technique may be extended to other quantizer types, such as logarithmical or hysteric, which may pose future study. As clarified in [7], the quantizer (4) has certain physical meanings and potential application prospects, such as a camera with zooming capability and a finite number of pixels that can be modeled as a quantizer (4).

Reference output model. The reference output signal \( y^*(t) \) is bounded and satisfies the condition

\[ Q(z)[y^*(t)] = 0, \]

where \( Q(z) \) is a monic polynomial of degree \( n_q \) with either nonrepeated zeros on the unit circle \( |z| = 1 \), or zeros inside the unit circle \( |z| < 1 \).

**Remark 1:** Unlike MRC, that only needs the reference signal being bounded, the pole placement-based tracking control technique further requires that the reference signal \( y^* \) satisfies the internal model (5) to achieve output tracking. As clarified in [20] and [21], the internal model condition (5) is necessary for the PPC design to achieve output tracking. Note that for a broad class of bounded time-varying signals, some appropriate \( Q(z) \) can be chosen to satisfy (5). For example, \( y^*(t) = a_1 \sin(\sigma t) + b_2 \cos(\sigma t) \), with \( \sigma \neq 0 \) and \( a_1^2 + b_2^2 \neq 0 \), \( Q(z) \) can be chosen as \( z^2 - 2z \cos \sigma + 1 \). For \( y^*(t) = a_1 \sin(\sigma_{1t}) + a_2 \sin(\sigma_{2t}) + b_1 \cos(\sigma_{1t}) + b_2 \cos(\sigma_{2t}) \), with \( \sigma_{1t} \neq 0, \sigma_{2t} \neq 0, \sigma_{1t} \neq \sigma_{2t} \), and \( a_1^2 + b_2^2 \neq 0, i = 1, 2 \), \( Q(z) \) can be chosen as \( (z^2 - 2z \cos \sigma_{1t} + 1)(z^2 - 2z \cos \sigma_{2t} + 1) \).

**Control objective.** For any given bounded \( y^*(t) \) satisfying (5), the control objective is to develop a quantized output feedback control law \( u(t) \) for the system model (1) ensuring that all closed-loop signals are bounded and \( y(t) - y^*(t) \) converges to a certain small residual set in certain finite time.

**Assumption.** To meet the control objective, the following assumption is required.

\( (A1): \) \( A(z)Q(z) \) and \( B(z) \) are coprime.

Assumption (A1) is a standard condition in classical PPC, with the finite-and-quantized output feedback PPC scheme still being effective under Assumption (A1). In the MRC design, \( B(z) \) must be stable. However, this paper does not require that \( B(z) \) is stable, i.e., the proposed method can deal with systems covering minimum-phase and non-minimum phase cases.

III. Finite-and-Quantized Output Feedback Control Design

This section develops a finite-and-quantized output feedback PPC scheme for LTI systems.

A. Fundamentals of Classical PPC

This part presents a key design equation in PPC design and gives a basic structure of the classic PPC law.

**Key design equation.** Choose a monic stable polynomial \( A^*(z) \) of degree \( 2n + n_q - 1 \), where \( n_q \) is the degree of \( Q(z) \) in (5). Under Assumption (A1), the following Diophantine equation can be solved

\[ C(z)Q(z)A(z) + D(z)B(z) = A^*(z), \]
with respect to $C(z)$ and $D(z)$ to find a unique solution of the form

$$C(z) = z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_1 z + c_0,$$

(7)

$$D(z) = d_{n+1}z^{n+1} + \cdots + d_1 z + d_0,$$

(8)

Utilizing (7) and (8),

$$C(z)Q(z) = z^{n+1} - \theta_{11}z^{n+2} - \cdots - \theta_{10}z - \theta_{10},$$

(9)

$$B(z)D(z) = \alpha_0^2z^{2n+2} + \alpha_1^2z^{2n+2} + \cdots + \alpha_6^2,$$  

(10)

In the PPC design, the design equation (6) must be solved to obtain the coefficients of $C(z)$ and $D(z)$. Under Assumption (A1), the solution of the design equation (6) is unique and can be obtained from an algebraic equation for any $A^*(z)$ of degree $2n + n_q - 1$. The corresponding proof is provided in [20] and [21].

**Classical PPC law.** From (8) and (9), the standard output feedback PPC law is given as [21]

$$u(t) = \theta_1^T\omega_1(z)[u(t) + \theta_2^T\omega_2(z)][g(t) - \theta_2^T\omega_2(z)[y^*(t)],$$

(11)

where

$$\theta_1^T = [\theta_1^T, \theta_1^T, \cdots, \theta_1^{(n+n_q)}] \in \mathbb{R}^{n+n_q-1},$$

(12)

$$\theta_2^T = [d_0, d_1, \cdots, d_{n+1}, \cdots, d_{n+n_q}] \in \mathbb{R}^{n+n_q},$$

(13)

$$\omega_1(z) = \left[ -\omega_0, -\omega_1, \cdots, -\omega_1 \right],$$

(14)

$$\omega_2(z) = \left[ -\omega_0, -\omega_1, \cdots, -\omega_1 \right].$$

(15)

**Remark 2:** From the control law (11), it can be proven that the tracking error $y(t) - y^*(t)$ converges to zero exponentially [21]. Since $A^*(z)$ in (6) is only required to be stable, a natural choice for $A^*(z)$ is $z^{2n+n_q-1}$. Particularly, when $A^*(z) = z^{2n+n_q-1}$, precise output tracking can be achieved in certain finite time. Specifically, the output feedback PPC law (11) with $A^*(z) = z^{2n+n_q-1}$ ensures $y(t_0 + 2n + n_q - 1) - y^*(t_0 + 2n + n_q - 1) = 0$, which can be concluded from the proof of Lemma 1 in the sequel. □

**B. Quantized-Output Feedback PPC Design**

The finite-and-quantized feedback PPC method has the following design details.

**Quantized-output feedback PPC law structure.** Motivated by the standard output feedback PPC law, the quantized-output feedback PPC law is designed as

$$u(t) = \theta_1^T\omega_1(z)[u(t) + \theta_2^T\omega_2(z)[\Delta q(y, \Delta)](t) - \theta_2^T\omega_2(z)[y^*(t)],$$

(16)

where $\Delta(t)$ is the sensitivity of $q$ in (4) to be designed later, and $\theta_1^*, \theta_2^*, \omega_1(z)$ and $\omega_2(z)$ are the same as in (11).

**Remark 3:** Note that $\theta_1^*$ and $\theta_2^*$ are constructed using the coefficients of the polynomials $D(z)$ and $C(z)Q(z)$. To construct both polynomials with $Q(z)$ being known, $C(z)$ and $D(z)$ are still needed. Note that $\{C(z), D(z)\}$ is the solution of equation (6). Thus, to solve equation (6), the main task is to calculate the inverse of the Sylvester matrix of $A(z)Q(z)$ and $B(z)$, which has an $2n + n_q$ dimension. Therefore, the computational complexity of the proposed approach is $O((2n + n_q)^3)$. □

**Tracking error equation.** Let the quantized error and the tracking error be

$$e(t) = y(t) - y^*(t),$$

(17)

$$s(y(t), \Delta(t)) = \Delta(t)q(y(t), \Delta(t)) - y(t),$$

(18)

respectively. Then, the following lemma specifies a tracking error equation, which is crucial for the sensitivity $\Delta(t)$ design and stability analysis.

**Lemma 1:** Considering $A^*(z) = z^{2n+n_q-1}$, the finite-and-quantized output feedback PPC law (16) applied to the system (1), ensures

$$e(t + 1) = \alpha^T\omega(z)[s(y, \Delta)](t), \forall t \geq t_0 + 2n + n_q - 2,$$

(19)

where $\alpha^* = -[\alpha_{2n+n_q-2}, \alpha_{2n+n_q-3}, \cdots, \alpha_0] \in \mathbb{R}^{2n+n_q-1}$ with $\alpha_i$ in (10) and $\omega(z) = [1, z, z^2, \cdots, z^{2n+n_q-2}]^T$. The proof of this lemma is provided in the Appendix. Equation (19) implies that the tracking error $y(t) - y^*(t)$ converges to zero in certain finite time when $A^*(z) = z^{2n+n_q-1}$ and using the exact output feedback, i.e., $s(y(t), \Delta(t)) = 0$. However, for the finite-and-quantized feedback PPC case, before designing $\Delta(t)$, it is unsure whether $s(y(t), \Delta(t))$ is bounded or not. Thus, the tracking error equation (19) does not imply the boundedness of $e(t)$. Next, it will be shown that with an appropriate choice of $\Delta(t)$, $s(y(t), \Delta(t))$ can be made small, and thus $e(t)$ can be made small.

**Specification of the control law.** Let

$$\lambda^* \triangleq \max\{\text{magnitudes of } |\lambda_i(A(z))|\},$$

(20)

where $\lambda_i(A(z)), i = 1, 2, \ldots, n$, denotes the zeros of $A(z)$ on the complex $z$-plane. Then, an appropriate $\lambda$ can be chosen such that $\lambda > \lambda^*$. Assuming that some appropriate integer $N \geq 1$ exists such that

$$M \geq \frac{c}{\gamma N - 1} + \frac{d}{\gamma N - 1} \left(1 + \frac{N}{M} \right) + \frac{1}{2},$$

(21)

where $c_0, \gamma$ are constants with $c_0 > 0$, $k \geq 0$, $0 < \gamma < 1$, and $c, d, t_1$ are defined as

$$e \triangleq \text{an upper bound of } \frac{1}{2}\|\alpha^*\|_1,$$

(22)

$$d \triangleq \text{an upper bound of } |y^*(t)|,$$

(23)

$$t_1 \triangleq \min \{t \geq t_0 + 1 : |q(y(t), \Delta(t))| \leq M - 1 \}. $$

(24)

Then, the finite-and-quantized output feedback PPC law is designed as

$$u(t) = \begin{cases} 0, & t \in [t_0, t_1), \\ \theta_1^T\omega_1(z)[u(t) + \theta_2^T\omega_2(z)[\Delta(t)q(y, \Delta(t))] + \theta_2^T\omega_2(z)[y^*(t)], & t \in [t_1, t_1 + 1), \\ \theta_1^T\omega_1(z)[u(t) + \theta_2^T\omega_2(z)[\Delta(t)q(y, \Delta(t))] + \theta_2^T\omega_2(z)[y^*(t)], & t \in [t_1 + 1, \infty), \\ \theta_1^T\omega_1(z)[u(t) + \theta_2^T\omega_2(z)[\Delta(t)q(y, \Delta(t))] + \theta_2^T\omega_2(z)[y^*(t)], & t \in [t_N, \infty), \\ \end{cases}$$

(25)

where $\Delta(t) = c_0 \gamma^t \epsilon$ for $t \in [t_0, t_1)$, and

$$\Delta(t) = c_0 \gamma^{t-1} \lambda^k(t_1 + 2n + n_q - 2), \quad i = 1, 2, \ldots, N,$$

(26)
with \( t_2 \triangleq t_1 + 2n + n_q - 1 \) and
\[
t_i \triangleq \min \left\{ t \geq t_{i-1} + 1 : |y(t)| \leq \frac{\Delta(t_i) + \Delta(t_{i-1}) + \frac{1}{2}}{\Delta(t_{i-1})} \right\}
\] (27)
for \( i = 3, 4, \cdots, N \).

Remark 4: When the system operates from \( t_0 \), the system input \( u(t) \) is zero and \( \Delta(t) = c_0 \lambda^t \). For \( u(t) = 0 \), the system model becomes \( A(z)y(t) = 0 \). In this case, the system output \( y(t) \) grows at most exponentially. Moreover, since \( \lambda > \lambda^* \) with \( \lambda^* \) in (20), it yields that \( \Delta(t) \) grows faster than \( |y(t)| \). Thus, regardless of \( y(t_0) \), \( \Delta(t_0) \) saturates or not, there always exists some finite \( t_1 \) such that \( \Delta(t_1) > |y(t_1)| \). Then, based on the definition of the quantizer, \( y(t_1) \) is not saturated. Hence, \( t_1 \) and \( \Delta(t_1) \) are both well-defined, which follows that \( t_2 \) is also well-defined and so is \( \Delta(t_2) \). In the proof of Theorem 1, it will be demonstrated that \( t_i, i = 3, 4, \ldots, N \) in (27) are all well-defined.

System performance analysis. With the finite-and-quantized output feedback PPC law (25), the main result is derived as follows.

**Theorem 1:** Under Assumption (A1), if inequality (21) holds, then the quantized-output feedback PPC law (25), applied to system (1) with any unmeasurable \( y(t_0) \in \mathbb{R} \), ensures that all closed-loop signals are bounded and the tracking error satisfies
\[
|e(t)| \leq c_0 \gamma^{N-1} \lambda^{k(t_1 + 2n + n_q - 2)}, \forall t \geq t_N + 2n + n_q - 1,
\] (28)
where \( c, c_0, k, \gamma \) are constants defined in (21) and (22).

The proof of this theorem is provided in the Appendix. Theorem 1 indicates that the proposed control law can achieve a bounded output tracking. Considering Theorem 1, one may raise the question of whether the tracking error \( e(t) \) can be made arbitrarily small. The following remark gives a positive answer to this question.

**Remark 5:** For any given constant \( \varepsilon > 0 \) and based on the proof of Theorem 1, one can verify that if \( M \) further satisfies the inequality
\[
M \geq \frac{cd}{\varepsilon} + \frac{c^2 c_0 \lambda^{k(t_1 + 2n + n_q - 2)}}{\varepsilon} + \frac{1}{2},
\] (29)
then the proposed control law (25) ensures that all closed-loop signals are bounded and the tracking error satisfies
\[
|e(t)| \leq \varepsilon, \forall t \geq t_N + 2n + n_q - 1.
\] (30)
This indicates that the proposed control law (25) achieves practical output tracking under a specified condition (29). The proof of the above conclusion is similar to Theorem 1 and thus is omitted to improve this paper’s readability.

So far, an analytical finite-and-quantized output feedback PPC law has been proposed with all signals and parameters being specified. Such a control law can achieve bounded or practical output tracking under different design conditions.

**IV. Simulation Study**

This section provides a representative example to illustrate the design procedure and validate the theoretical results.

**Simulation model.** Consider the system model
\[
A_r(z)y(t) = B_r(z)u(t),
\]
where \( A_r(z) = \left( z + \frac{1}{2} \right)(z - 2), B_r(z) = z + 2 \). This model is unstable as an unstable pole \( z = 2 \) exists. In addition, there exists an unstable zero \( z = -2 \) and thus the simulation model is non-minimum phase. The reference output is chosen as \( y^*(t) = 20 \sin \left( \frac{\pi}{5} t \right) - 2 \cos \left( \frac{\pi}{5} t \right) \).

Given equation \( Q(z)[y^*(t)] = 0 \), it follows that \( Q(z) = z^2 + 1 \).

**Specification of \( \theta^*_1 \) and \( \theta^*_2 \).** Note that \( \theta^*_1 \) and \( \theta^*_2 \) in the quantized-output feedback PPC law are the same as in the classic output feedback PPC law (16). Thus, given that \( A_r(z), B_r(z), Q(z) \) and the design equation (6) have known coefficients, it follows that \( \theta^*_1 = [-0.5, -0.5, 0, 0, 0, 0, 0, 0, 0, 0]^T \), \( \theta^*_2 = [-0.5, -0.5, 0, 0, 0, 0, 0, 0, 0, 0]^T \), and \( \alpha^* = [-0.5, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T \).

**Quantized output feedback PPC law.** The following parameters are defined: from (20), \( \lambda = 2.1 \); from (27), \( c = 6 \); and from (21), \( d = 25 \). On the basis of (25), the constant parameters \( c_0, M, k, \gamma \) are chosen as \( c_0 = 1, M = 4001, k = 1 \), and \( \gamma = \frac{1}{2} \). Then, from (21), \( N = 10 \). Thus, the control law can be specified from (25).

**Simulation results.** This paper considers two scenarios \( y(0) = 4500 \) and \( y(0) = 30 \) to verify the proposed control method independent of the initial conditions. For \( y_0 = 4500 \), Fig. 1 illustrates the response of the system output \( y(t) \) versus the reference output \( y^*(t) \), revealing that the tracking performance is satisfactory when \( t > 24 \). Fig. 2 depicts the response of the quantized output feedback PPC law, while Fig. 3 presents the response of the quantizer sensitivity \( \Delta(t) \). From the latter figure, it can be seen that \( t_1 = 1, t_2 = 6, t_3 = 8, t_4 = 9, t_5 = 11, t_6 = 13, t_7 = 15, t_8 = 17, t_9 = 18, \) and \( t_{10} = 20 \), with the evolution of \( \Delta(t) \) exactly matching the
In summary, the simulation results validate the effectiveness of the proposed method and its independence from the initial conditions. Moreover, recalling that \( A_q(z) \) and \( B_q(z) \) are not stable, the simulation results also verify the validity of the proposed method for the non-minimum phase system.

V. CONCLUDING REMARKS

This paper presents a finite-and-quantized output feedback version of the standard PPC law and demonstrates its effectiveness in controlling general discrete-time LTI systems under relaxed design conditions. The finite-and-quantized output feedback PPC law is analytically constructed and only relies on the condition that \( A(z)Q(z) \) and \( B(z) \) are coprime. The suggested scheme provides a feasible method to effectively control the non-minimum phase systems subject to output quantization and saturation.

Although the developed scheme poses an appealing solution, it would be interesting to consider the following problems: (i) how to modify the proposed control method so that asymptotic output tracking can be achieved; and (ii) how to realize adaptive output tracking control for systems with unknown parameters based on the proposed control method.

APPENDIX

Proof of Lemma 1. Let \( A^*(z) = z^{2n+n_q-1} \) and \( z \neq 0 \). The key design equation (6) can be written in the form

\[
C(z)Q(z)A(z) + D(z)B(z) = z.
\]  

By operating both sides of (31) on \( y(t) \) and \( y^*(t) \),

\[
C(z)Q(z)A(z)\frac{[y](t)}{z^{2n+n_q-2}} + D(z)B(z)\frac{[y](t)}{z^{2n+n_q-2}} = y(t + 1),
\]  

(32)

With the reference system (5) and (33), it follows that

\[
D(z)B(z)\frac{[y^*](t)}{z^{2n+n_q-2}} = y^*(t + 1).
\]  

With (16), it yields

\[
u(t) = \left( \frac{\theta_{q0}}{z^{n+n_q} - 1} + \frac{\theta_{q1}}{z^{2n+n_q} - 1} + \cdots + \frac{\theta_{q(n+n_q-2)}}{z^{n_q-1}} \right) [y](t) + \left( \frac{d_0}{z^{n+n_q} - 1} + \frac{d_1}{z^{2n+n_q} - 1} + \cdots + \frac{d_{n_q-1}}{z^{n_q-1}} \right) [\Delta y](t) + \left( \frac{d_0}{z^{n+n_q} - 1} + \frac{d_1}{z^{2n+n_q} - 1} + \cdots + \frac{d_{n_q-1}}{z^{n_q-1}} \right) [y^*](t)
\]

\[
= \left( 1 - \frac{C(z)Q(z)}{z^{2n+n_q-1}} \right) [u](t) + \frac{D(z)}{z^{2n+n_q-1}} [\Delta y](t) + \frac{D(z)}{z^{2n+n_q-1}} [y^*](t).
\]

Thus, it follows

\[
C(z)Q(z)[u](t) = D(z)[y^*](t) - D(z)[\Delta y](t).
\]  

(34)

Using the system model (1), it yields

\[
y(t + 1) = \frac{C(z)Q(z)B(z)}{z^{2n+n_q-2}} [u](t) + \frac{D(z)B(z)}{z^{2n+n_q-2}} [y](t)
\]

\[
y^*(t + 1) + \frac{D(z)B(z)}{z^{2n+n_q-2}} [y - \Delta y](t).
\]  

(35)

Therefore, it follows that

\[
e(t + 1) = -\frac{D(z)B(z)}{z^{2n+n_q-1}} [s](t),
\]

which implies \( e(t + 2n+n_q-1) = -D(z)B(z)[s](t), \forall t \geq 0 \).
Finally, it follows that $y(t) \in L^\infty$ from $y^*(t) \in L^\infty$. Then, by operating both sides of (6) on $u(t)$ with $A^*(z) = z^{2n+na-1}$, yields $C_{\OO}(z)A(z)+D(z)B(z)[y(t)] = u(t)$. Combined with (1) and (34), it can be derived that $u(t) = \frac{C_{\OO}(z)A(z)}{A^*(z)}[y(t)] + \frac{D(z)B(z)}{A^*(z)}[y(t)]$. Since $y^*(t)$ and $s(y(t), \Delta(t))$ are bounded and $A^*(z) = z^{2n+na-1}$ is stable, it yields that $u(t) \in L^\infty$.

REFERENCES