# A Pole Placement-Based Output Tracking Control Scheme by Finite-and-Quantized Output Feedback

Yuchun Xu, Yanjun Zhang, *Member, IEEE*, and Ji-Feng Zhang, *Fellow, IEEE*

***Abstract*—This letter proposes a finite-and-quantized output feedback output tracking control scheme for possibly non-minimum phase discrete-time linear systems that are subject to output quantization and saturation. An analytical pole placement-based control law is proposed by solely utilizing the finite-and-quantized output and the external reference output. The closed-loop stability and output tracking analysis are essentially different from the classical pole placement method. It needs to overcome some new technical issues caused by finite-and-quantized output feedback, such as how to realize closed-loop stability while restricting the finite quantization of the output measurement. This letter demonstrates that by appropriately designing the quantizer's sensitivity, the proposed control law ensures all closed-loop signals are bounded, and the output tracking error converges to a certain residual set of the origin within a certain finite time, regardless of the existence of unstable zeros and poles in the control systems. Particularly, the residual set can be arbitrarily small under a specified design condition. Finally, a representative example validates the proposed control scheme.**

***Index Terms*—Discrete-time, finite time, output tracking, pole placement control, quantized-output feedback.**

## I. Introduction

During the past two decades, the problems of control systems subject to quantized and/or saturated constraints have attracted significant attention in the control community. Generally, classic state and output feedback control methods cannot be directly applied to control systems subject to quantization and saturation measurements, especially for the finite quantized case. Additionally, practical control systems suffer from measurement errors due to the sensor's limitations in accuracy and magnitude. Moreover, compared to exact feedback control designs, the finite-and-quantized feedback control technique is more robust in anti-disturbances. Therefore, it is of theoretical and practical significance to develop new methods that improve the system performance for control systems subject to quantization and saturation problems.

To date, tremendous development has been made in quantized feedback control theory and applications. In the 1960s, [1] introduced the quantized control idea to reduce the computational burden in optimum designs. Since then, various quantized control methods have been developed, such as [2]–[6]. The quantizers in quantized feedback control systems are generally divided into static and dynamic, with the former having fixed quantized levels and quantized errors. Nevertheless, solely relying on static quantization is challenging to afford global or semi-global convergence. Therefore, [7] developed a dynamic quantized method, which, compared to the static, has an adjustable parameter called "the sensitivity" of the quantizer. Both quantizer types have been widely used in various control system classes, e.g., deterministic control systems [8], stochastic control systems [9], multi-agent consensus or formation [10], [11], and network control systems [12], [13]. In particular, the stabilization and tracking control problems have been extensively studied utilizing quantized control [14]–[16]. Furthermore, the saturation problem has also been considered in quantized control systems, where [17] studied the control problem of the systems subject to saturation and proposed several constructive anti-windup methods, and [18] investigated the robust stabilization problem of uncertain linear systems under saturated state feedback.

Recently, [19] established a basic framework for tracking control of discrete-time linear time-invariant (LTI) systems by using finite-and-quantized output feedback. Moreover, a new model reference control (MRC) scheme was developed by utilizing finite-and-quantized output feedback. However, such a control scheme is only suitable for controlling minimum-phase systems, still suffering from the fundamental control problem: how to effectively control a general class of discrete-time LTI systems covering the minimum and non-minimum phase

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Yuchun Xu and Ji-Feng Zhang are with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and also with the School of Mathematics Sciences, University of Chinese Academy of Sciences, Beijing 100149, China (e-mail: xuyuchun0115@163.com; jif@iss.ac.cn).

Yanjun Zhang is with the School of Automation, Beijing Institute of Technology, Beijing 100081, China (e-mail: yanjun@bit.edu.cn).

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cases. In [20] and [21], the authors demonstrated that the pole placement control (PPC) method achieves an appealing output tracking via the exact state or output feedback. However, it is an open research case whether a finite-and-quantized output feedback version of the classic PPC scheme is still valid. To our knowledge, such a problem has not been addressed yet. Compared with the classic PPC method, the finite-and-quantized feedback case suffers from some new technical problems, such as realizing closed-loop stability and achieving output tracking while restricting the finite quantization of the output measurement. Hence, this letter aims to address these concerns systematically. The main contributions of this letter are as follows:

(i) Providing an affirmative "yes" answer to the problem of whether a finite-and-quantized output feedback version of the classic PPC law is still valid. In particular, a quantized-and-saturated output feedback version of the classic PPC law is analytically constructed.

(ii) Compared with the existing literature, the proposed control method has distinctive characteristics. First, it ensures that the output tracking error converges to a residual set of the origin within a certain finite time. Second, global convergence of the output tracking is achieved in the sense that the proposed control law is independent of the system's initial conditions. Finally, the controlled plant is allowed to have unstable poles and zeros, i.e., the proposed control method is effective for both minimum-phase and non-minimum phase systems.

The remainder of this letter is as follows. Section II introduces the system model and control problems to be addressed. Section III reviews the fundamental PPC law and presents the details of the finite-and-quantized output feedback PPC scheme. Section IV presents some simulation examples, and finally, Section V concludes this letter.

## II. Problem Statement

This section presents the system model and the problems investigated in this letter.

*System model:* Consider the following discrete-time single-input and single-output (SISO) LTI system:

$$A(z)[y](t) = B(z)[u](t), \quad t \geq t_0, \tag{1}$$

where $t_0$ is the initial moment of the system operation and $A(z), B(z)$ are polynomials with constant coefficients of degree $n$ and $n-1$, respectively, i.e.,

$$A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \tag{2}$$

$$B(z) = b_{n-1}z^{n-1} + b_{n-2}z^{n-2} + \cdots + b_1 z + b_0. \tag{3}$$

In this letter, $z$ and $z^{-1}$ denote the forward and backward shift operators, i.e., $z[x](t) = x(t+1)$ and $z^{-1}[x](t) = x(t-1)$, where $t \in \{0, 1, 2, 3, \ldots,\}$, $x(t) \triangleq x(tT)$ for a sampling period $T > 0$ and $x(t)$ denotes any signal of any finite dimension. For the system model (1), $y(t)$ cannot be measured accurately and one can only acquire its finite and quantized values denoted as $q(y(t), \Delta(t))$, where $q$ is the quantizer to be given.

*Dynamic quantizer:* Let $X(t) \in \mathbb{R}$ be any signal on $\mathbb{R}$. Then, this letter's quantizer is specified as

$$q(X(t), \Delta(t)) = \begin{cases} M, & \text{if } X(t) > (M+\frac{1}{2})\Delta(t), \\ [\frac{X(t)}{\Delta(t)} + \frac{1}{2}], & \text{if } -(M+\frac{1}{2})\Delta(t) < X(t) \leq (M+\frac{1}{2})\Delta(t), \\ -M, & \text{if } X(t) \leq -(M+\frac{1}{2})\Delta(t), \end{cases} \tag{4}$$

where $\Delta(t) \neq 0$ depends on $t$ and is called the sensitivity of $q$, $M$ is a positive integer, and $[X(t)] \triangleq \max\{k \in \mathbb{Z} : k < X(t)\}$. Although the proposed method does not have any restrictive conditions on the used quantizer, this letter relies on the quantizer proposed in [7]. In this sense, the suggested technique may be extended to other quantizer types, such as logarithmical or hysteretic, which may pose future study. As clarified in [7], the quantizer (4) has certain physical meanings and potential application prospects, such as a camera with zooming capability and a finite number of pixels that can be modeled as a quantizer (4).

*Reference output model:* The reference output signal $y^*(t)$ is bounded and satisfies the condition

$$Q(z)[y^*](t) = 0, \tag{5}$$

where $Q(z)$ is a monic polynomial of degree $n_q$ with either nonrepeated zeros on the unit circle $|z| = 1$, or zeros inside the unit circle $|z| < 1$.

*Remark 1:* Unlike MRC, that only needs the reference signal being bounded, the pole placement-based tracking control technique further requires that the reference signal $y^*$ satisfies the internal model (5) to achieve output tracking. As clarified in [20] and [21], the internal model condition (5) is necessary for the PPC design to achieve output tracking. Note that for a broad class of bounded time-varying signals, some appropriate $Q(z)$ can be chosen to satisfy (5). For example, for $y^*(t) = a\sin(\sigma t) + b\cos(\sigma t)$, with $\sigma \neq 0$ and $a^2 + b^2 \neq 0$, $Q(z)$ can be chosen as $z^2 - 2z\cos\sigma + 1$. For $y^*(t) = a_1\sin(\sigma_1 t) + a_2\sin(\sigma_2 t) + b_1\cos(\sigma_1 t) + b_2\cos(\sigma_2 t)$, with $\sigma_1 \neq 0, \sigma_2 \neq 0, \sigma_1 \neq \sigma_2$ and $a_i^2 + b_i^2 \neq 0$, $i = 1, 2$, $Q(z)$ can be chosen as $(z^2 - 2z\cos\sigma_1 + 1)(z^2 - 2z\cos\sigma_2 + 1)$.

*Control objective:* For any given bounded $y^*(t)$ satisfying (5), the control objective is to develop a quantized output feedback control law $u(t)$ for the system model (1) ensuring that all closed-loop signals are bounded and $y(t) - y^*(t)$ converges to a certain small residual set in certain finite time.

*Assumption:* To meet the control objective, the following assumption is required.

**(A1):** $A(z)Q(z)$ and $B(z)$ are coprime.

Assumption (A1) is a standard condition in classical PPC, with the finite-and-quantized output feedback PPC scheme still being effective under Assumption (A1). In the MRC design, $B(z)$ must be stable. However, this letter does not require that $B(z)$ is stable, i.e., the proposed method can deal with systems covering minimum-phase and non-minimum phase cases.

## III. Finite-and-Quantized Output Feedback Control Design

This section develops a finite-and-quantized output feedback PPC scheme for LTI systems.

### A. Fundamentals of Classical PPC

This part presents a key design equation in PPC design and gives a basic structure of the classic PPC law.

*Key design equation:* Choose a monic stable polynomial $A^*(z)$ of degree $2n+n_q-1$, where $n_q$ is the degree of $Q(z)$ in (5). Under Assumption (A1), the following Diophantine equation can be solved

$$C(z)Q(z)A(z)+D(z)B(z)=A^*(z), \tag{6}$$

with respect to $C(z)$ and $D(z)$ to find a unique solution of the form

$$C(z)=z^{n-1}+c_{n-2}z^{n-2}+\cdots+c_1z+c_0, \tag{7}$$
$$D(z)=d_{n_q+n-1}z^{n_q+n-1}+\cdots+d_1z+d_0. \tag{8}$$

Utilizing (7) and (8),

$$C(z)Q(z)=z^{n+n_q-1}-\theta^*_{1(n+n_q-2)}z^{n+n_q-2}-\cdots-\theta^*_{11}z-\theta^*_{10}, \tag{9}$$
$$B(z)D(z)=\alpha^*_{2n+n_q-2}z^{2n+n_q-2}+\alpha^*_{2n+n_q-3}z^{2n+n_q-3}+\cdots+\alpha^*_0. \tag{10}$$

In the PPC design, the design equation (6) must be solved to obtain the coefficients of $C(z)$ and $D(z)$. Under Assumption (A1), the solution of the design equation (6) is unique and can be obtained from an algebraic equation for any $A^*(z)$ of degree $2n+n_q-1$. The corresponding proof is provided in [20] and [21].

*Classical PPC law:* From (8) and (9), the standard output feedback PPC law is given as [21]

$$u(t)=\theta_1^{*T}\omega_1(z)[u](t)+\theta_2^{*T}\omega_2(z)[y](t)-\theta_2^{*T}\omega_2(z)[y^*](t), \tag{11}$$

where

$$\theta_1^*=[\theta^*_{10},\theta^*_{11},\ldots,\theta^*_{1(n+n_q-2)}]^T\in\mathbb{R}^{n+n_q-1}, \tag{12}$$
$$\theta_2^*=-[d_0,d_1,\ldots,d_{n+n_q-2},d_{n+n_q-1}]^T\in\mathbb{R}^{n+n_q}, \tag{13}$$
$$\omega_1(z)=[z^{-n-n_q+1},z^{-n-n_q+2},\ldots,z^{-1}]^T, \tag{14}$$
$$\omega_2(z)=[z^{-n-n_q+1},z^{-n-n_q+2},\ldots,z^{-1},1]^T. \tag{15}$$

*Remark 2:* From the control law (11), it can be proven that the tracking error $y(t)-y^*(t)$ converges to zero exponentially [21]. Since $A^*(z)$ in (6) is only required to be stable, a natural choice for $A^*(z)$ is $z^{2n+n_q-1}$. Particularly, when $A^*(z)=z^{2n+n_q-1}$, precise output tracking can be achieved in certain finite time. Specifically, the output feedback PPC law (11) with $A^*(z)=z^{2n+n_q-1}$ ensures $y(t_0+2n+n_q-1)-y^*(t_0+2n+n_q-1)=0$, which can be concluded from the proof of Lemma 1 in the sequel.

## B. Quantized-Output Feedback PPC Design

The finite-and-quantized feedback PPC method has the following design details.

*Quantized-output feedback PPC law structure:* Motivated by the standard output feedback PPC law, the quantized-output feedback PPC law is designed as

$$u(t)=\theta_1^{*T}\omega_1(z)[u](t)+\theta_2^{*T}\omega_2(z)[\Delta q(y,\Delta)](t) - \theta_2^{*T}\omega_2(z)[y^*](t), \tag{16}$$

where $\Delta(t)$ is the sensitivity of $q$ in (4) to be designed later, and $\theta_1^*,\theta_2^*,\omega_1(z)$ and $\omega_2(z)$ are the same as in (11).

*Remark 3:* Note that $\theta_1^*$ and $\theta_2^*$ are constructed using the coefficients of the polynomials $D(z)$ and $C(z)Q(z)$. To construct both polynomials with $Q(z)$ being known, $C(z)$ and $D(z)$ are still needed. Note that $\{C(z),D(z)\}$ is the solution of equation (6). Thus, to solve equation (6), the main task is to calculate the inverse of the Sylvester matrix of $A(z)Q(z)$ and $B(z)$, which has an $2n+n_q$ dimension. Therefore, the computational complexity of the proposed approach is $O((2n+n_q)^3)$.

*Tracking error equation:* Let the quantized error and the tracking error be

$$e(t)=y(t)-y^*(t), \tag{17}$$
$$s(y(t),\Delta(t))=\Delta(t)q(y(t),\Delta(t))-y(t), \tag{18}$$

respectively. Then, the following lemma specifies a tracking error equation, which is crucial for the sensitivity $\Delta(t)$ design and stability analysis.

*Lemma 1:* Considering $A^*(z)=z^{2n+n_q-1}$, the finite-and-quantized output feedback PPC law (16) applied to the system (1), ensures

$$e(t+1)=\alpha^{*T}\omega(z)[s(y,\Delta)](t), \forall t\geq t_0+2n+n_q-2, \tag{19}$$

where $\alpha^*=-[\alpha^*_{2n+n_q-2},\alpha^*_{2n+n_q-3},\ldots,\alpha^*_0]^T\in\mathbb{R}^{2n+n_q-1}$ with $\alpha_i^*$ in (10) and $\omega(z)=[1,z^{-1},\ldots,z^{-2n-n_q+2}]^T$.

The proof of this lemma is provided in the Appendix. Equation (19) implies that the tracking error $y(t)-y^*(t)$ converges to zero in certain finite time when $A^*(z)=z^{2n+n_q-1}$ and using the exact output feedback, i.e., $s(y(t),\Delta(t))=0$. However, for the finite-and-quantized feedback PPC case, before designing $\Delta(t)$, it is unsure whether $s(y(t),\Delta(t))$ is bounded or not. Thus, the tracking error equation (19) does not imply the boundedness of $e(t)$. Next, it will be shown that with an appropriate choice of $\Delta(t)$, $s(y(t),\Delta(t))$ can be made small, and thus $e(t)$ can be made small.

*Specification of the control law:* Let

$$\lambda^*\triangleq\max\{\text{magnitudes of } |\lambda_i(A(z))|\}, \tag{20}$$

where $\lambda_i(A(z))$, $i=1,2,\ldots,n$, denotes the zeros of $A(z)$ on the complex $z-$plane. Then, an appropriate $\lambda$ can be chosen such that $\lambda>\lambda^*$. Assuming that some appropriate integer $N\geq 1$ exists such that

$$M\geq\frac{c}{\gamma^{N-1}}+\frac{d}{c_0\gamma^{N-1}\lambda^{k(t_1+2n+n_q-2)}}+\frac{1}{2}, \tag{21}$$

where $c_0,k,\gamma$ are constants with $c_0>0,\ k\geq 1,\ 0<\gamma<1$, and $c,d,t_1$ are defined as

$$c\triangleq\text{an upper bound of }\frac{1}{2}\|\alpha^*\|_1, \tag{22}$$
$$d\triangleq\text{an upper bound of }|y^*(t)|, \tag{23}$$
$$t_1\triangleq\min\{t\geq t_0+1:|q(y(t),\Delta(t))|\leq M-1\}. \tag{24}$$

Then, the finite-and-quantized output feedback PPC law is designed as

$$u(t)=\begin{cases}0, & t\in[t_0,t_1),\\ \theta_1^{*T}\omega_1(z)[u](t) & \\ +\theta_2^{*T}\omega_2(z)[\Delta(t_i)q(y,\Delta(t_i))](t) & \\ -\theta_2^{*T}\omega_2(z)[y^*](t), & t\in[t_i,t_{i+1}),\\ & i=1,\ldots,N-1,\\ \theta_1^{*T}\omega_1(z)[u](t) & \\ +\theta_2^{*T}\omega_2(z)[\Delta(t_N)q(y,\Delta(t_N))](t) & \\ -\theta_2^{*T}\omega_2(z)[y^*](t), & t\in[t_N,\infty),\end{cases} \tag{25}$$

where $\Delta(t) = c_0\lambda^{kt}$ for $t \in [t_0, t_1)$, and

$$\Delta(t_i) = c_0\gamma^{i-1}\lambda^{k(t_1+2n+n_q-2)},\ i = 1, 2, \ldots, N, \tag{26}$$

with $t_2 \triangleq t_1 + 2n + n_q - 1$ and

$$t_i \triangleq \min\{t \geq t_{i-1} + 1 : |q(y(t), \Delta(t))| \leq \frac{c\Delta(t_{i-1}) + |y^*(t)|}{\Delta(t_{i-1})} + \frac{1}{2}\} \tag{27}$$

for $i = 3, 4, \ldots, N$.

*Remark 4:* When the system operates from $t_0$, the system input $u(t)$ is zero and $\Delta(t) = c_0\lambda^{kt}$. For $u(t) = 0$, the system model becomes $A(z)[y](t) = 0$. In this case, the system output $y(t)$ grows at most exponentially. Moreover, since $\lambda > \lambda^*$ with $\lambda^*$ in (20), it yields that $\Delta(t)$ grows faster than $|y(t)|$. Thus, regardless of $q(y(t_0), \Delta(t_0))$ saturates or not, there always exists some finite $t_1$ such that $\Delta(t_1) > |y(t_1)|$. Then, based on the definition of the quantizer, $q(y(t_1), \Delta(t_1))$ is not saturated. Hence, $t_1$ and $\Delta(t_1)$ are both well-defined, which follows that $t_2$ is also well-defined and so is $\Delta(t_2)$. In the proof of Theorem 1, it will be demonstrated that $t_i$, $i = 3, 4, \ldots, N$ in (27) are all well-defined.

*System performance analysis:* With the finite-and-quantized output feedback PPC law (25), the main result is derived as follows.

*Theorem 1:* Under Assumption (A1), if inequality (21) holds, then the quantized-output feedback PPC law (25), applied to system (1) with any unmeasurable $y(t_0) \in \mathbb{R}$, ensures that all closed-loop signals are bounded and the tracking error satisfies

$$|e(t)| \leq cc_0\gamma^{N-1}\lambda^{k(t_1+2n+n_q-2)}, \forall t \geq t_N + 2n + n_q - 1, \tag{28}$$

where $c, c_0, k, \gamma$ are constants defined in (21) and (22).

The proof of this theorem is provided in the Appendix. Theorem 1 indicates that the proposed control law can achieve a bounded output tracking. Considering Theorem 1, one may raise the question of whether the tracking error $e(t)$ can be made arbitrarily small. The following remark gives a positive answer to this question.

*Remark 5:* For any given constant $\varepsilon > 0$ and based on the proof of Theorem 1, one can verify that if $M$ further satisfies the inequality

$$M \geq \frac{cd}{\varepsilon} + \frac{c^2c_0\lambda^{k(t_1+2n+n_q-2)}}{\varepsilon} + \frac{1}{2}, \tag{29}$$

then the proposed control law (25) ensures that all closed-loop signals are bounded and the tracking error satisfies

$$|e(t)| \leq \varepsilon,\ \ \forall t \geq t_N + 2n + n_q - 1. \tag{30}$$

This indicates that the proposed control law (25) achieves practical output tracking under a specified condition (29). The proof of the above conclusion is similar to Theorem 1 and thus is omitted to improve this letter's readability.

So far, an analytical finite-and-quantized output feedback PPC law has been proposed with all signals and parameters being specified. Such a control law can achieve bounded or practical output tracking under different design conditions.

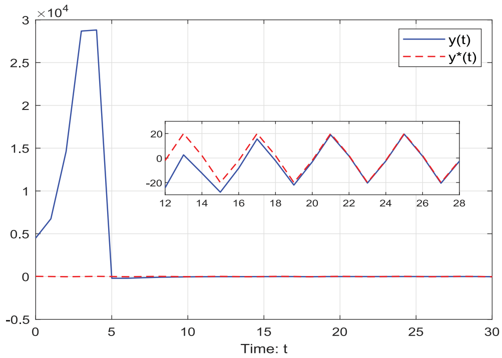


Fig. 1. $y(t)$ v.s. $y^*(t)$ ($y_0 = 4500$).

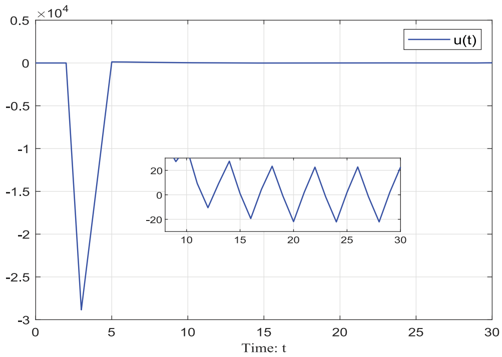


Fig. 2. Response of the control law ($y_0 = 4500$).

## IV. Simulation Study

This section provides a representative example to illustrate the design procedure and validate the theoretical results.

*Simulation model:* Consider the system model $A_r(z)[y](t) = B_r(z)[u](t)$, where $A_r(z) = (z + \frac{1}{2})(z - 2), B_r(z) = z + 2$. This model is unstable as an unstable pole $z = 2$ exists. In addition, there exists an unstable zero $z = -2$ and thus the simulation model is non-minimum phase. The reference output is chosen as $y^*(t) = 20\sin(\frac{\pi}{2}t) - 2\cos(\frac{\pi}{2}t)$. Given equation $Q(z)[y^*](t) = 0$, it follows that $Q(z) = z^2 + 1$.

*Specification of $\theta_1^*$ and $\theta_2^*$:* Note that $\theta_1^*$ and $\theta_2^*$ in the quantized-output feedback PPC law are the same as in the classic output feedback PPC law (16). Thus, given that $A_r(z), B_r(z), Q(z)$ and the design equation (6) have known coefficients, it follows that $\theta_1^* = [-\frac{14}{15}, -1, -\frac{14}{15}]^T$, $\theta_2^* = [-\frac{7}{15}, -\frac{29}{30}, -\frac{4}{15}, -\frac{17}{30}]^T$ and $\alpha^* = -[\frac{17}{30}, \frac{7}{5}, \frac{3}{2}, \frac{12}{5}, \frac{14}{15}]^T$.

*Quantized output feedback PPC law:* The following parameters are defined: from (20), $\lambda = 2.1$; from (27), $c = 6$; and from (21), $d = 25$. On the basis of (25), the constant parameters $c_0, M, k, \gamma$ are chosen as $c_0 = 1, M = 4001, k = 1$, and $\gamma = \frac{1}{2}$. Then, from (21), $N = 10$. Thus, the control law can be specified from (25).

*Simulation results:* This letter considers two scenarios $y(0) = 4500$ and $y(0) = 30$ to verify the proposed control method independent of the initial conditions. For $y_0 = 4500$, Fig. 1 illustrates the response of the system output $y(t)$ versus the reference output $y^*(t)$, revealing that the tracking performance is satisfactory when $t > 24$. Fig. 2 depicts the response of the quantized output feedback PPC law, while

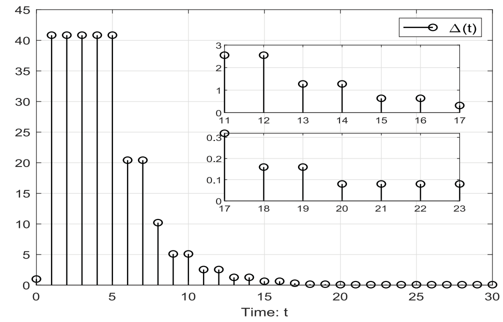


Fig. 3. Response of $\Delta(t)$ ($y_0 = 4500$).

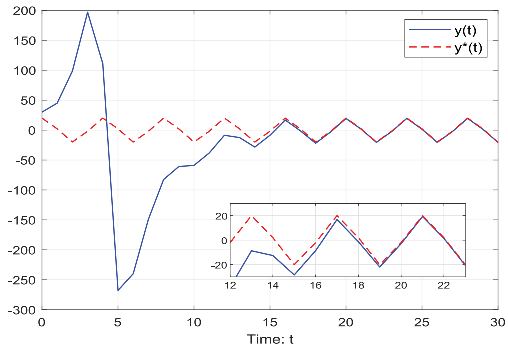


Fig. 4. $y(t)$ v.s. $y^*(t)$ ($y_0 = 30$).

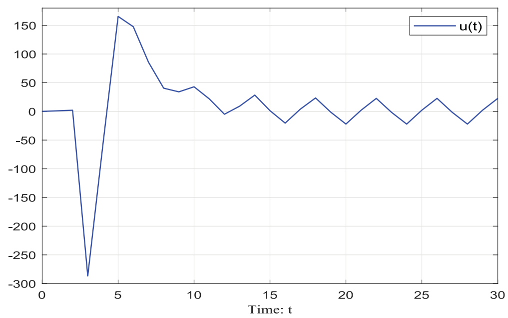


Fig. 5. Response of the control law ($y_0 = 30$).

Fig. 3 presents the response of the quantizer sensitivity $\Delta(t)$. From the latter figure, it can be seen that $t_1 = 1$, $t_2 = 6$, $t_3 = 8$, $t_4 = 9$, $t_5 = 11$, $t_6 = 13$, $t_7 = 15$, $t_8 = 17$, $t_9 = 18$, and $t_{10} = 20$, with the evolution of $\Delta(t)$ exactly matching the theoretical results. For the $y_0 = 30$ case, Figs. 4-6 highlight that the proposed method achieves satisfactory results.

In summary, the simulation results validate the effectiveness of the proposed method and its independence from the initial conditions. Moreover, recalling that $A_s(z)$ and $B_s(z)$ are not stable, the simulation results also verify the validity of the proposed method for the non-minimum phase system.

## V. Conclusion

This letter presents a finite-and-quantized output feedback version of the standard PPC law and demonstrates its effectiveness in controlling general discrete-time LTI systems under relaxed design conditions. The finite-and-quantized output feedback PPC law is analytically constructed and only replies on the condition that $A(z)Q(z)$ and $B(z)$ are coprime. The suggested scheme provides a feasible method to effectively control the non-minimum phase systems subject to output quantization and saturation.

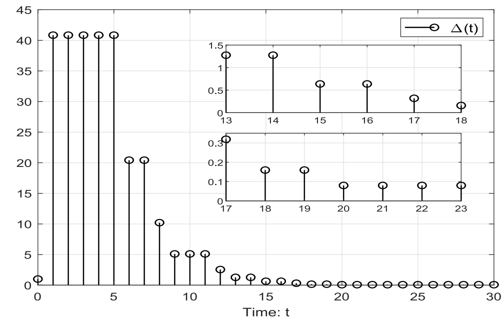


Fig. 6. Response of $\Delta(t)$ ($y_0 = 30$).

Although the developed scheme poses an appealing solution, it would be interesting to consider the following problems: (i) how to modify the proposed control method so that asymptotic output tracking can be achieved; and (ii) how to realize adaptive output tracking control for systems with unknown parameters based on the proposed control method.

## Appendix
## Proof of Lemma 1

Let $A^*(z) = z^{2n+n_q-1}$ and $z \neq 0$. The key design equation (6) can be written in the form

$$\frac{C(z)Q(z)A(z)}{z^{2n+n_q-2}} + \frac{D(z)B(z)}{z^{2n+n_q-2}} = z. \tag{31}$$

By operating both sides of (31) on $y(t)$ and $y^*(t)$,

$$\frac{C(z)Q(z)A(z)}{z^{2n+n_q-2}}[y](t) + \frac{D(z)B(z)}{z^{2n+n_q-2}}[y](t) = y(t+1), \tag{32}$$

$$\frac{C(z)Q(z)A(z)}{z^{2n+n_q-2}}[y^*](t) + \frac{D(z)B(z)}{z^{2n+n_q-2}}[y^*](t) = y^*(t+1). \tag{33}$$

With the reference system (5) and (33), it follows that $\frac{D(z)B(z)}{z^{2n+n_q-2}}[y^*](t) = y^*(t+1)$. With (16), it yields

$$\begin{aligned}
u(t) &= \Big(\frac{\theta_{10}^*}{z^{n+n_q-1}} + \frac{\theta_{11}^*}{z^{n+n_q-2}} + \cdots + \frac{\theta_{1(n+n_q-2)}^*}{z}\Big)[u](t) \\
&\quad - \Big(\frac{d_0}{z^{n+n_q-1}} + \frac{d_1}{z^{n+n_q-2}} + \cdots + \frac{d_{n+n_q-1}}{z^0}\Big)[\Delta q(y, \Delta)](t) \\
&\quad + \Big(\frac{d_0}{z^{n+n_q-1}} + \frac{d_1}{z^{n+n_q-2}} + \cdots + \frac{d_{n+n_q-1}}{z^0}\Big)[y^*](t) \\
&= \Big(1 - \frac{C(z)Q(z)}{z^{n+n_q-1}}\Big)[u](t) + \frac{D(z)}{z^{n+n_q-1}}[y^*](t) \\
&\quad - \frac{D(z)}{z^{n+n_q-1}}[\Delta q(y, \Delta)](t).
\end{aligned}$$

Thus, it follows

$$C(z)Q(z)[u](t) = D(z)[y^*](t) - D(z)[\Delta q(y, \Delta)](t). \tag{34}$$

Using the system model (1), it yields

$$\begin{aligned}
y(t+1) &= \frac{C(z)Q(z)B(z)}{z^{2n+n_q-2}}[u](t) + \frac{D(z)B(z)}{z^{2n+n_q-2}}[y](t) \\
&= y^*(t+1) + \frac{D(z)B(z)}{z^{2n+n_q-2}}[y - \Delta q(y, \Delta)](t).
\end{aligned} \tag{35}$$

Therefore, it follows that $e(t+1) = -\frac{D(z)B(z)}{z^{2n+n_q-2}}[s(y,\Delta)](t)$, which implies $e(t+2n+n_q-1) = -D(z)B(z)[s(y,\Delta)](t), \forall t \geq t_0$. It further follows that $e(t+1) = \alpha^{*T}\omega(z)[s(y,\Delta)](t), \forall t \geq t_0+2n+n_q-2$, which completes the proof. $\nabla$

*Proof of Theorem 1:* When the system operates from $t_0$, $u(t) = 0$, a well-defined $t_1$ can be obtained.

When $t = t_1$, together with the definition of $t_1$ in (24) and the structure of quantizer in (4), it follows that $|y(t_1)| \leq c_0\lambda^{kt_1}(M-\frac{1}{2})$. For $t \geq t_1$, the control law is changed from $u(t) = 0$ to (16) with $\Delta(t) = \Delta(t_1)$. Thus, based on the fact that $c_0\lambda^{kt}$ grows faster than $y(t)$, it yields $|y(t_1+j)| \leq c_0\lambda^{k(t_1+j)}(M-\frac{1}{2}) \leq \Delta(t_1)(M-\frac{1}{2})$ for $j = 0, 1, \ldots, 2n+n_q-2$ and $\Delta(t_1) = c_0\lambda^{k(t_1+2n+n_q-2)}$, which implies that $y(t_1+j), j = 0, 1, \ldots, 2n+n_q-2$, all are not saturated. Then, for $t_1 \leq t \leq t_1+2n+n_q-2$, it follows $|s(y(t),\Delta(t))| = |\Delta(t_1)[\frac{y(t)}{\Delta(t_1)}+\frac{1}{2}]-y(t)| \leq \frac{\Delta(t_1)}{2}$. Thus, from Lemma 1 it follows that $|y(t_1+2n+n_q-1) - y^*(t_1+2n+n_q-1)| = |\alpha^*\omega(z)[s(y,\Delta)](t_1+2n+n_q-2)| \leq |\frac{1}{2}\alpha^*\Delta(t_1)| \leq c\Delta(t_1)$. Then, it yields $|y(t_1+2n+n_q-1)| \leq c\Delta(t_1) + |y^*(t_1+2n+n_q-1)|$. Thus, from (21), it follows $|y(t_1+2n+n_q-1)| \leq \Delta(t_1)(c+\frac{|y^*(t_1+2n+n_q-1)|}{\Delta(t_1)}) \leq \Delta(t_1)(M-\frac{1}{2})$, which implies that $q(y(t_1+2n+n_q-1), \Delta(t_1+2n+n_q-1))$ is not saturated. Repeating the above procedure, it can be further verified that $q(y(t),\Delta(t))$ is not saturated for all $t \geq t_1+2n+n_q-1$. Thus, it can be concluded that, if $\Delta(t)$ is chosen as $\Delta(t_1)$ in (16) for all $t \geq t_1$, $q(y(t),\Delta(t))$ will never saturate and $e(t)$ satisfies $|e(t)| \leq c\Delta(t_1), \forall t \geq t_1+2n+n_q-1$.

Then, let $t_2 = t_1+2n+n_q-1$. When $t = t_2$, the control law changes to (16) with $\Delta(t) = \gamma\Delta(t_1)$. Then, $|y(t_2)| \leq c\Delta(t_1) + |y^*(t_2)| = \gamma\Delta(t_1)(\frac{c}{\gamma}+\frac{|y^*(t_2)|}{\gamma\Delta(t_1)}) \leq \Delta(t_2)(M-\frac{1}{2})$, which implies $q(y(t_2),\Delta(t_2))$ is not saturated. From Lemma 1, it yields $|y(t_2+1) - y^*(t_2+1)| = |\alpha^*\omega(z)[s(y,\Delta)](t_2)| \leq c\Delta(t_1)$. Thus, from (21) it follows $|y(t_2+1)| \leq c\Delta(t_1) + |y^*(t_2+1)| = \gamma\Delta(t_1)(\frac{c}{\gamma}+\frac{|y^*(t_2+1)|}{\gamma\Delta(t_1)}) \leq \Delta(t_2)(M-\frac{1}{2})$, which implies that $y(t_2+1)$ is not saturated. By repeating the above procedure, it can be proven that for all $t \geq t_2$, $q(y(t),\Delta(t))$ is not saturated, i.e., $|y(t)| \leq \Delta(t_2)(M-\frac{1}{2}), t \geq t_2$. Then, from Lemma 1, for $t \geq t_2+2n+n_q-1$, it follows that $|e(t)| = |y(t)-y^*(t)| = |\alpha^*\omega(z)[s(y,\Delta)](t-1)| \leq c\Delta(t_2) = c\gamma\Delta(t_1)$, which implies $|y(t)| \leq c\Delta(t_2) + |y^*(t)|, t \geq t_2+2n+n_q-1$. Thus, there exists a well-defined number $t_3$ as defined in (27).

When $t = t_3$, the control law changes to (16) with $\Delta(t) = \Delta(t_3)$ and using Lemma 1 it yields $|y(t_3) - y^*(t_3)| \leq c\Delta(t_1)$, $|y(t_3)| \leq c\Delta(t_1) + |y^*(t_3)| = \gamma^2\Delta(t_1)(\frac{c}{\gamma^2}+\frac{|y^*(t_3)|}{\gamma^2\Delta(t_1)}) \leq \Delta(t_3)(M-\frac{1}{2})$, which implies that $q(y(t_3),\Delta(t_3))$ is not saturated. Thus, using Lemma 1 again, it follows that $q(y(t_3+1),\Delta(t_3+1))$ is not saturated. Then, it further follows that $q(y(t_3+2),\Delta(t_3+2))$ is not saturated, neither do $q(y(t_3+3),\Delta(t_3+3)), q(y(t_3+4),\Delta(t_3+4)), \ldots$ Therefore, it can be concluded that, if the control law is chosen as (16) with $\Delta(t) = \Delta(t_3)$ for $t \geq t_3$, $q(y(t),\Delta(t))$ will never saturate and from Lemma 1 it follows that $e(t)$ satisfies $|e(t)| \leq c\gamma^2\Delta(t_1), t \geq t_3+2n+n_q-1$. Repeating the above procedure, $t_4, \ldots, t_N$ can be obtained and a sequence $\{\Delta(t_i)\}_{1\leq i\leq N}$ can be derived. Moreover, the control law (25) can ensure that $e(t)$ satisfies (28).

Finally, it follows that $y(t) \in L^\infty$ from $y^*(t) \in L^\infty$. Then, by operating both sides of (6) on $u(t)$ with $A^*(z) = z^{2n+n_q-1}$, yields $\frac{C(z)Q(z)A(z)+D(z)B(z)}{A^*(z)}[u](t) = u(t)$. Combined with (1) and (34), it can be derived that $u(t) = \frac{A(z)D(z)}{A^*(z)}[y^*](t) - \frac{A(z)D(z)}{A^*(z)}[\Delta q(y,\Delta)](t) + \frac{D(z)A(z)}{A^*(z)}[y](t) = \frac{A(z)D(z)}{A^*(z)}[s(y,\Delta)](t) + \frac{A(z)D(z)}{A^*(z)}[y^*](t)$. Since $y^*(t)$ and $s(y(t),\Delta(t))$ are bounded and $A^*(z) = z^{2n+n_q-1}$ is stable, it yields that $u(t) \in L^\infty$.

## References


[1] R. Larson, "Optimum quantization in dynamic systems," *IEEE Trans. Autom. Control*, vol. AC-12, no. 2, pp. 162–168, Apr. 1967.

[2] K. Li and J. Baillieul, "Robust quantization for digital finite communication bandwidth (DFCB) control," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1573–1584, Sep. 2004.

[3] D. Liberzon, "Quantization, time delays, and nonlinear stabilization," *IEEE Trans. Autom. Control*, vol. 51, no. 7, pp. 1190–1195, Jul. 2006.

[4] H. Gao and T. Chen, "A new approach to quantized feedback control systems," *Automatica*, vol. 44, no. 2, pp. 534–542, 2008.

[5] M. L. Corradini and G. Orlando, "Robust quantized feedback stabilization of linear systems," *Automatica*, vol. 44, no. 9, pp. 2458–2462, 2008.

[6] X. Chang, J. Xiong, Z. Li, and J. H. Park, "Quantized static output feedback control for discrete-time systems," *IEEE Trans. Ind. Informat.*, vol. 14, no. 8, pp. 3426–3435, Aug. 2018.

[7] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[8] Z. Liu, F. Wang, Y. Zhang, and C. L. P. Chen, "Fuzzy adaptive quantized control for a class of stochastic nonlinear uncertain systems," *IEEE Trans. Cybern.*, vol. 46, no. 2, pp. 524–534, Feb. 2016.

[9] L. Sheng, Z. Wang, W. Wang, and F. E. Alsaadi, "Output-feedback control for nonlinear stochastic systems with successive packet dropouts and uniform quantization effects," *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 47, no. 7, pp. 1181–1191, Jul. 2017.

[10] Z. Zhang, L. Zhang, F. Hao, and L. Wang, "Leader-following consensus for linear and Lipschitz nonlinear multiagent systems with quantized communication," *IEEE Trans. Cybern.*, vol. 47, no. 8, pp. 1970–1982, Aug. 2017.

[11] Y. Meng, T. Li, and J.-F. Zhang, "Coordination over multi-agent networks with unmeasurable states and finite-level quantization," *IEEE Trans. Autom. Control*, vol. 62, no. 9, pp. 4647–4653, Sep. 2017.

[12] R. Yang, P. Shi, G.-P. Liu, and H. Gao, "Network-based feedback control for systems with mixed delays based on quantized and drop out compensation," *Automatica*, vol. 47, no. 12, pp. 2805–2809, 2011.

[13] K. Liu, E. Fridman, and K. H. Johansson, "Dynamic quantization of uncertain linear networked control systems," *Automatica*, vol. 59, pp. 248–255, Sep. 2015.

[14] M. Fu and L. Xie, "Finite-Level quantized feedback control for linear systems," *IEEE Trans. Autom. Control*, vol. 54, no. 5, pp. 1165–1170, May 2009.

[15] D. Fan, X. Zhang, Y. Chang, and X. Lu, "Global practical tracking via disturbance rejection control for uncertain nonlinear systems with quantized input and output," *Sci. China Inf. Sci.*, vol. 65, no. 1, pp. 199–201, 2022.

[16] C.-Y. Liu, Z.-Y. Sun, Q. Meng, and W. Sun, "Robust control of high-order nonlinear systems with unknown measurement sensitivity," *Sci. China Inf. Sci.*, vol. 64, no. 6, pp. 1–3, 2021.

[17] S. Galeani, S. Tarbouriech, M. C. Turner, and L. Zaccarian, "A tutorial on modern anti-windup design," *Eur. J. Control*, vol. 15, nos. 3–4, pp. 418–440, 2009.

[18] Y. Chen, S. Fei, and Y. Li, "Robust stabilization for uncertain saturated time-delay systems: A distributed-delay-dependent polytopic approach," *IEEE Trans. Autom. Control*, vol. 62, no. 7, pp. 3455–3460, Jul. 2017.

[19] Y. J. Zhang, J.-F. Zhang, X.-K. Liu, and Z. Liu, "Quantized-output feedback model reference control of discrete-time linear systems," *Automatica*, vol. 137, Mar. 2022, Art. no. 110027.

[20] P. Ioannou and J. Sun, *Robust Adaptive Control*. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[21] G. Tao, *Adaptive Control Design and Analysis*. Hoboken, NJ, USA: Wiley, 2003.